The relation between finite differences in time and the Chebyshev polynomial recursion

Reynam Pestana[1], Paul Stoffa[2] and Adriano Santos[3]

[1]CPGG/UFBA and INCT-GP/CNPq, Salvador, Bahia, Brazil

[2]UT Austin, Texas, USA.

[3]INCT-GP/CNPq, Salvador, Bahia, Brazil

82nd Annual Meeting SEG
4-9 November 2012

Las Vegas, Nevada USA

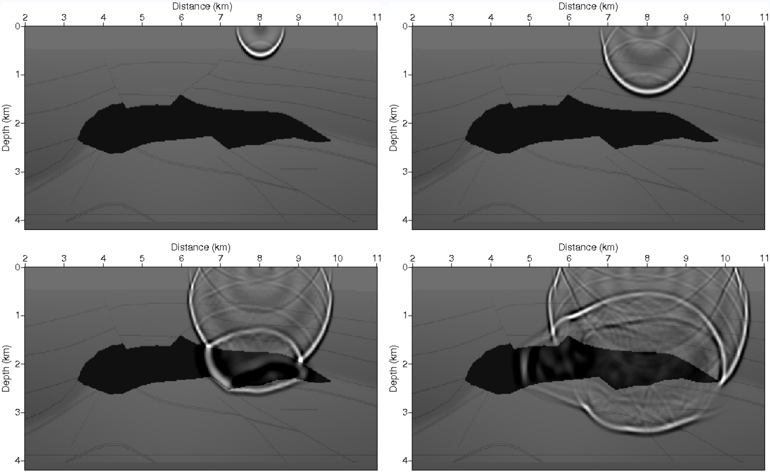
Motivation

Pestana and Stoffa, 2010, Time evolution of wave equation using rapid expansion method (REM), Geophysics, vol. 75 no. 4.

Remarks:

- Numerical implementation of the Chebyshev polynomial recursion have shown wavelike character results;
- We have shown that the REM reduces to the same equations used for the second-order finite-difference time approximation (asymptotic approximation for the Bessel function) .

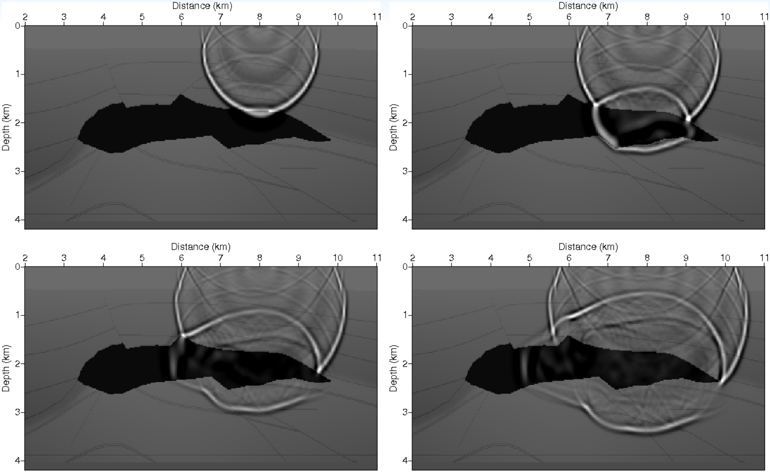
Chebyshev polynomials



Top-left Bot-left Top-right Bot-right

Q_{200} (top left); Q_{400} (top right); Q_{600} (bottom left); Q_{800} (bottom right)

Snapshots computed by REM



▸ Top-left ▸ Bot-left ▸ Top-right ▸ Bot-right

Snapshosts at: 1.0 s (top left); 1.2 s (top right); 1.4 s (bottom left); 1.6 s (bottom right)

Acoustic wave equation

$$\frac{\partial^2 P(\mathbf{x},t)}{\partial t^2} = -L^2 P(\mathbf{x},t) \tag{1}$$

with $-L^2 = c^2(\mathbf{x})\nabla^2$

Formal solution of equation 1, with the initial condition

$$\begin{cases} \frac{\partial P(\mathbf{x},t)}{\partial t}\Big|_{t=0} = \dot{P}_0 \\ P(\mathbf{x},t=0) = P_0 \end{cases}$$

$$P(\mathbf{x},t) = \cos(Lt)\,P_0 + L^{-1}\sin(Lt)\,\dot{P}_0 \tag{2}$$

One-step solution by REM

$$P(\mathbf{x},t) + P(\mathbf{x},-t) = 2\cos(Lt)\,P_0 \tag{3}$$

The rapid expansion method (REM)

The cosine function is given by (Kosloff et. al, 1989)

$$\cos(Lt) = \sum_{k=0}^{M} C_{2k}\, J_{2k}(R\,t)\, Q_{2k}\left(\frac{iL}{R}\right) \tag{4}$$

where $R = \pi c_{max}\sqrt{(\frac{1}{\Delta x})^2 + (\frac{1}{\Delta z})^2}$ and $M > R\,t$ (Tal-Ezer, 1987).

Chebyshev polynomials recursion is given by:

$$Q_{k+2}(x) = (4x^2 + 2)\, Q_k(x) \, - \, Q_{k-2}\,(x)$$

with the initial values:

$$\begin{cases} Q_0(x) = 1 \\ Q_2(x) = 1 + 2x^2 \end{cases}$$

REM - recursive solution

Using the wave equation solution we have:

$$P(t+\Delta t) + P(t-\Delta t) = 2\cos(L\Delta t)\,P(t) \tag{5}$$

Taking the Taylor series expansion of $\cos(L\Delta t)$.

$$\begin{cases} (1 - \frac{(L\Delta t)^2}{2}) & - \text{ Second order} \\ (1 - \frac{(L\Delta t)^2}{2} + \frac{(L\Delta t)^4}{24}) & - \text{ Fourth order} \end{cases}$$

We obtain the standard finite-difference schemes:

$$\frac{P(t+\Delta t) - 2\,P(t) + P(t-\Delta t)}{\Delta t^2} = c^2\nabla^2\,P(t) \tag{6}$$

$$\frac{P(t+\Delta t) - 2\,P(t) + P(t-\Delta t)}{\Delta t^2} = \left(c^2\nabla^2 + \frac{c^4\Delta t^2}{12}\,\nabla^4\right)\,P(t) \tag{7}$$

Finite difference solution - special case of REM

Using the REM approximation, the wave equation solution is given by:

$$P(t+\Delta t)+P(t-\Delta t)=2\sum_{k=0}^{M} C_{2k}\,J_{2k}(z)\,Q_{2k}(w)\,P(t) \qquad (8)$$

where $w=\frac{i\,L}{R}$ and $z=R\,\Delta t$.

Asymptotically the Bessel function behaves as

$$J_k(z)\approx\frac{1}{k!}\left(\frac{z}{2}\right)^k\approx\frac{1}{\sqrt{2\pi k}}\left(\frac{ez}{2k}\right)^k \qquad (9)$$

for $k\longrightarrow\infty$, hence for $k\gg R\Delta t$ the Bessel function decay exponentially and the series can be truncated with negligible error.

Finite difference solution - special case of REM

Considering only the terms in $(w\,z)^n$ for $n = 0, 2, 4\cdot$, we obtain:

$$
\begin{aligned}
P(t+\Delta t) + P(t-\Delta t) &= 2\left(1 + \frac{z^2}{2}\,w^2 + \frac{z^4}{24}\,w^4 + \frac{z^6}{720}\,w^6\right. \\
&+ \left.\frac{z^8}{40320}\,w^8 + \cdots\right) P(t) \qquad (10)
\end{aligned}
$$

Now, considering only the terms up to Δt^2, and substituting w by $\frac{iL}{R}$ and $z = R\,\Delta t$, we get:

Second-order finite difference in time scheme.

$$P(t+\Delta t) - 2\,P(t) + P(t-\Delta t) = -\Delta t^2\,L^2\,P(t) \qquad (11)$$

Finite difference solution - special case of REM

In the same way, for the 4th order approximation we have:

$$\frac{1}{\Delta t^2}\left[P(t+\Delta t) - 2\,P(t) + P(t-\Delta t) - \frac{\Delta t^4}{12}\,L^4\,P(t)\right] = -\,L^2\,p(t) \tag{12}$$

Then,

$$-\,L^4\,P(t) = L^2\,\frac{\partial^2 P}{\partial t^2} = -\frac{\partial^2}{\partial t^2}(-L^2 P) = -\frac{\partial^4 P}{\partial t^4} \tag{13}$$

So the L^4 operator term has been replaced by $\partial^4/\partial t^4$.

Thus, REM is also a Lax-Wendroff scheme - higher-order time derivatives are replaced by spatial derivatives

Two coupled first order wave equation

The acoustic wave equation:

$$\frac{1}{c^2}\frac{\partial^2 P}{\partial t^2} = \frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial z^2} \tag{14}$$

Setting $Q = \partial_t P$,

Two coupled first order equation

$$\frac{\partial}{\partial t}\left(\begin{array}{c} P \\ Q \end{array}\right) = \left(\begin{array}{cc} 0 & 1 \\ c^2\nabla^2 & 0 \end{array}\right)\left(\begin{array}{c} P \\ Q \end{array}\right) \tag{15}$$

A compact notation

$$\frac{\partial V}{\partial t} = G\, V \tag{16}$$

where $V = \left(\begin{array}{c} P \\ Q \end{array}\right)$ and $G = \left(\begin{array}{cc} 0 & 1 \\ c^2\nabla^2 & 0 \end{array}\right)$

First order differential wave equation

The pressure wavefield $\hat{P}$ is a complex wavefield defined as (Zhang and Zhang, 2009)

$$\hat{P}(x, z, t) = P(x, z, t) + iH[P(x, z, t)] \tag{17}$$

where $H[\cdot]$ is the Hilbert transform operator.

The complex pressure wavefield $\hat{P}$ satisfies the following first-order partial equation in the time direction

$$\frac{\partial \hat{P}}{\partial t} = \Phi \hat{P}. \tag{18}$$

where Φ is a pseudodifferential operator and is represented by

$$\phi = i\, c(x, z) \sqrt{k_x^2 + k_z^2} \tag{19}$$

Chebyshev polynomials

Let's introduce the Chebyshev polynomials

$$T_n(x) = \cos(n\theta), \quad \text{where} \quad x = \cos\theta \tag{20}$$

The trigonometric recursion

$$2\cos(\theta)\cos(n\theta) = \cos((n+1)\theta) + \cos((n-1)\theta)$$

implies the recursion

$$T_{n+1}(x) = 2\,x\,T_n(x) - T_{n-1}(x) \tag{21}$$

with $T_0 = 1$ and $T_1(x) = x$

The modified Chebyshev polynomials satisfy the recurrence

$$Q_{n+1}(x) = 2\,x\,Q_n(x) + Q_{n-1}(x) \tag{22}$$

where $Q_n(x) = i^n\,T_n(-ix)$; Again, the recursion is initiated by:
$Q_0(x) = 1 \; \textit{and} \; Q_1(x) = x$

Finite difference and Chebyshev polynomials

Centered finite difference scheme:

$$\frac{\partial \hat{P}^n}{\partial t} = \frac{\hat{P}^{n+1} - \hat{P}^{n-1}}{2\Delta t} \tag{23}$$

We have:

$$\frac{\partial \hat{P}}{\partial t} = \Phi \hat{P} \quad \Longrightarrow \quad \hat{P}_{n+1} = 2\Delta t\, \Phi\, \hat{P}_n + \hat{P}_{n-1}$$

Comparing with the Chebyshev polynomial recursion

$$Q_{n+1}(x) = 2\, x\, Q_n(x) + Q_{n-1}(x)$$

we notice that

$$\hat{P}_{n+1} = Q_{n+1}(\Delta t\, \Phi)\hat{P}_0 \qquad \text{for} \quad n = 1, 2, \cdots \tag{24}$$

In this way, the finite difference wavefields are just the Chebyshev polynomials in $\Delta t\, \Phi$ acting on the initial (injected source) wavefield $\hat{P}_0$.

Finite difference and Chebyshev polynomials

Since $Q_n(x)$ is bounded on $(-1, 1)$ for all $n \geq 0$ and unbounded for any x not belonging to $(-1, 1)$. The finite-difference scheme is stable if and only if

$$\Delta t < \frac{1}{R}$$

where R, for the 2D case, is given by

$$R = \pi \, c_{max} \sqrt{\left(\frac{1}{dx}\right)^2 + \left(\frac{1}{dz}\right)^2} \tag{25}$$

If $\Delta x = \Delta z$, the stability limit is given by:

$$\alpha = \frac{c_{max}\, \Delta t}{\Delta x} < 0.2$$

which is the stability condition for the pseudospectral method as recommended by Kosloff and Baysal (1982).

Laplacian computation

Laplacian operator - 2D case

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial z^2}$$

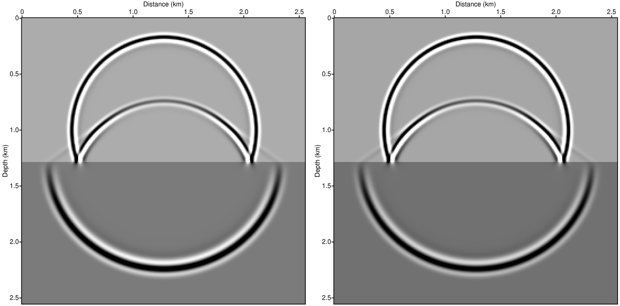
We can use the Fourier space for evaluating the Laplacian - using forward and inverse discrete Fourier transforms (DFT), as:

$$\nabla^2 P = DFT^{-1}[-||\vec{k}||^2 \, DFT(P)]$$

where $\vec{k} = (k_x, k_z)$ is the 2D wave number vector.

Impulse response - two layer model

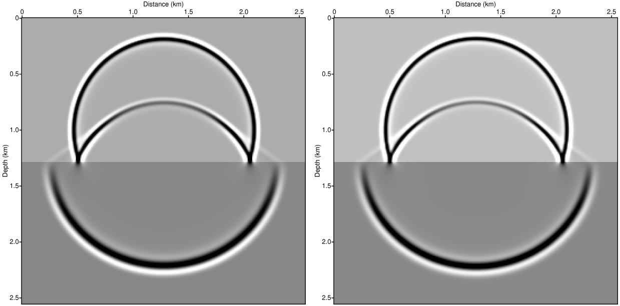
wavefield snapshot at t=1.26 ms



Impulse response for the two layer model based on the solution of:
Coupled wave equation system (left); First-order wave equation; (right)

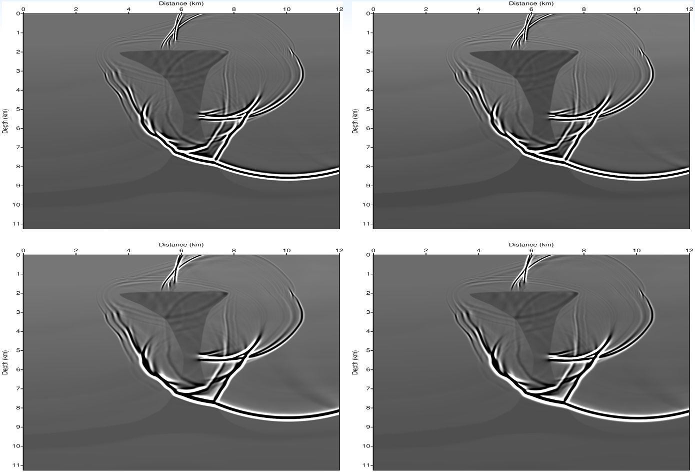
Impulse response - two layer model

wavefield snapshot at t=1.26 ms



Impulse response for the two layer model based on the solution of: Modified Chebyshev polynomials (left); The REM (right)

Impulse response for the BP salt model



Top-left Bot-left Top-right Bot-right

Coupled wave equation system; (top left); First-order wave equation; (top right); Modified Chebyshev polynomials (bottom left); and the REM (bottom right)

Conclusions

- The coupled wave equation system finite difference approximation in time follows exactly the recursion of the modified Chebyshev polynomials in $\mathbf{A}\Delta t$.
- We have shown the same connection with the solution of the first-order wave equation by finite differences, but with the Chebyshev polynomials in $\Phi\Delta t$ where Φ is a pseudodifferential operator.
- The modified Chebyshev polynomials in the REM has the same connection with the finite difference method of the full wave equation using a second order time-difference approximation.

Conclusions

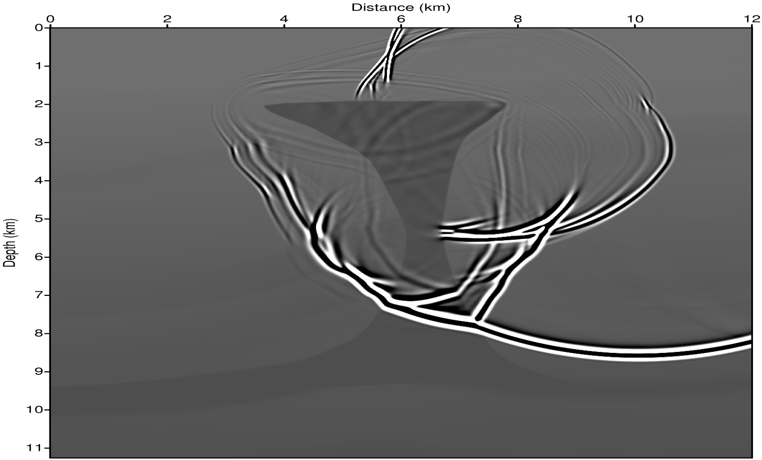
- We have also derived the stability condition of these method exploring the range of validity of the Chebyshev polynomials.
- FD methods through numerical examples with the Chebyshev polynomial recursion can be used to generate the propagation of seismic waves which are free of numerical dispersion and unconditionally stable.

Acknowledgments

- This research was supported by CNPq and INCT-GP/CNPq.
- The facility support from CPGG/UFBA is also acknowledged.

Two coupled first order wave equation result

wavefield snapshot at t=2.9 s

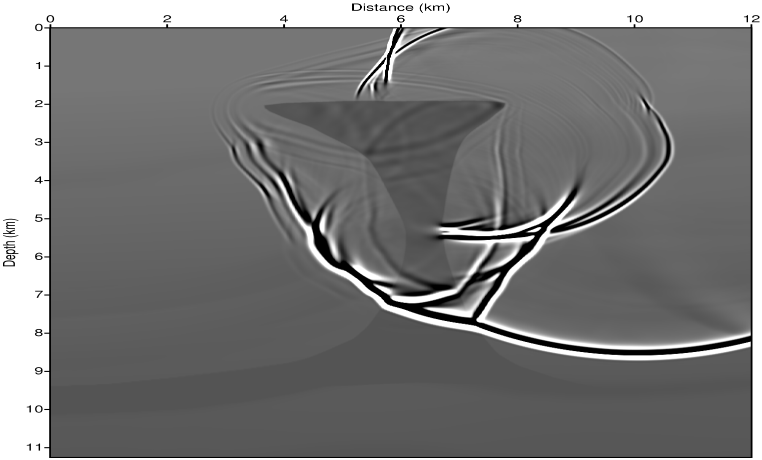


▸ Back

$\Delta t = 0.645\,ms$ and $f_{max} = 30\,Hz$

First order differential wave equation result

wavefield snapshot at t=2.9 s

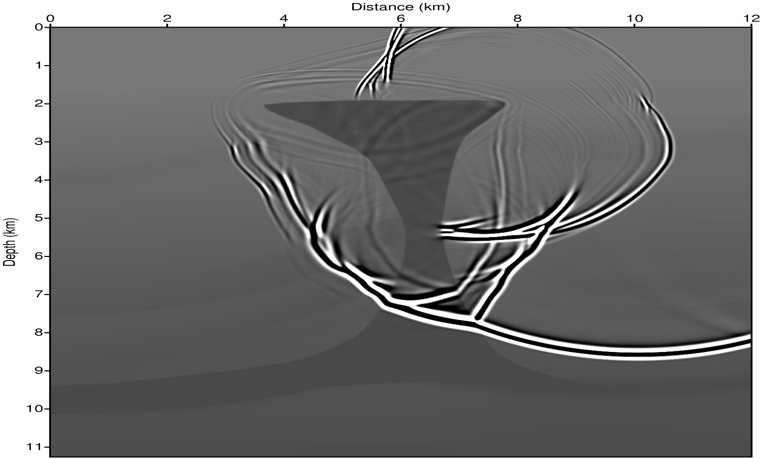


Back

$\Delta t = 0.645\, ms$ and $f_{max} = 30\, Hz$

Modified chebyshev polynomials result

wavefield snapshot at t=2.9 s

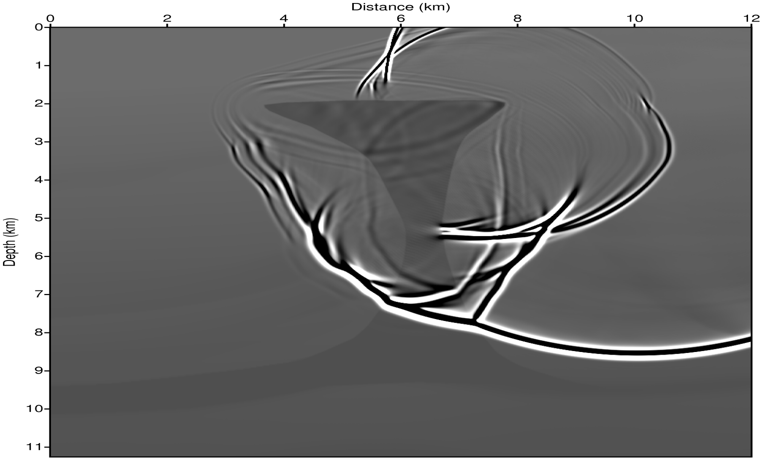


Back

$\Delta t = 1.29\,ms$ (only the even terms); $f_{max} = 30\,Hz$.

Rapid expansiom method (REM) result

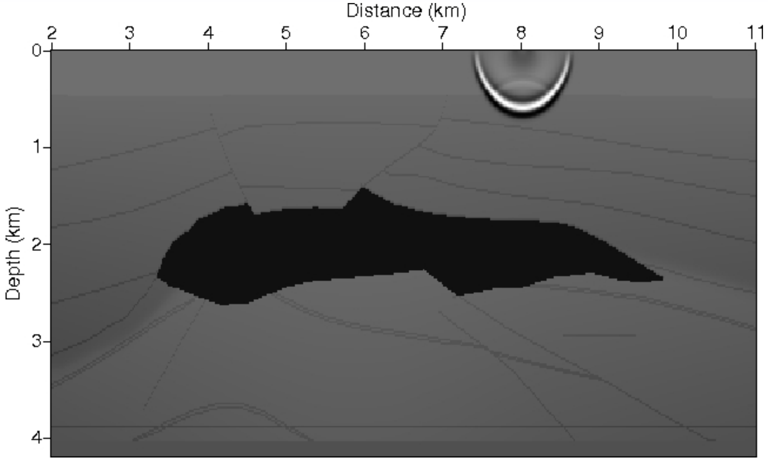
wavefield snapshot at t=2.9 s



▸ Back

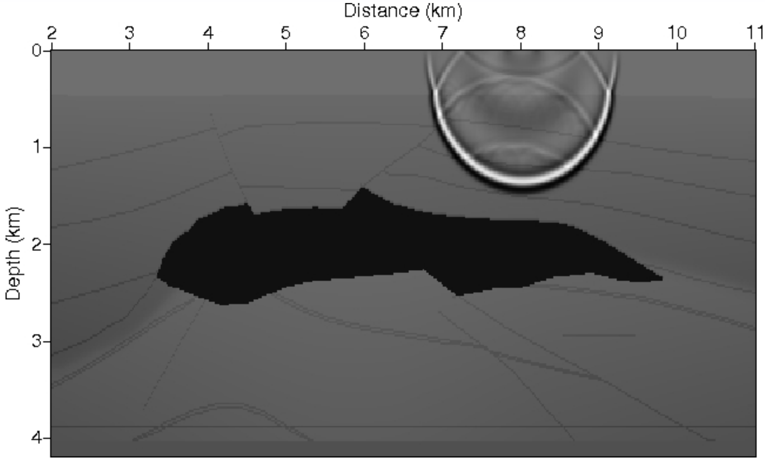
Using a time stepping 7 times larger than the other three methods;

Chebyshev polynomial Q_{200}



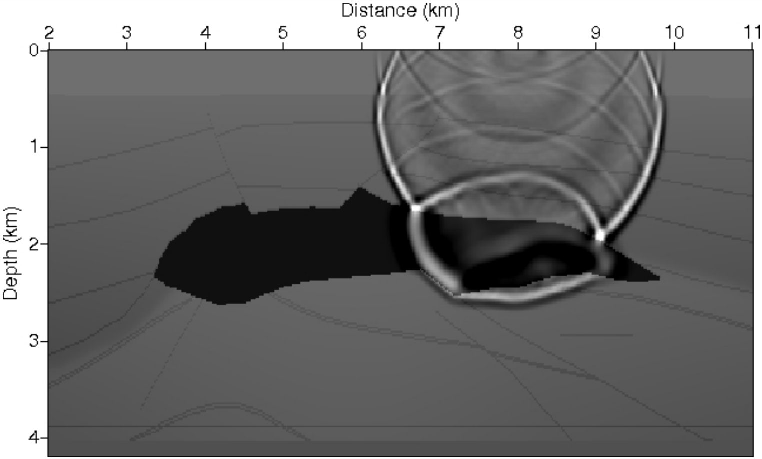
Back

Chebyshev polynomial Q_{400}



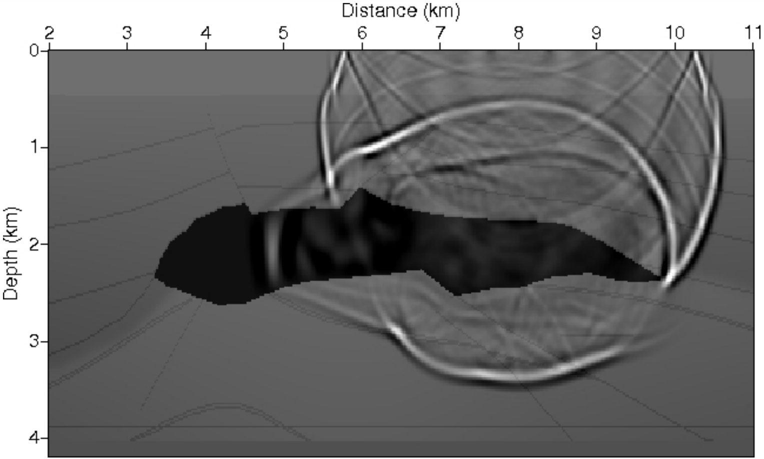
▸ Back

Chebyshev polynomial Q_{600}



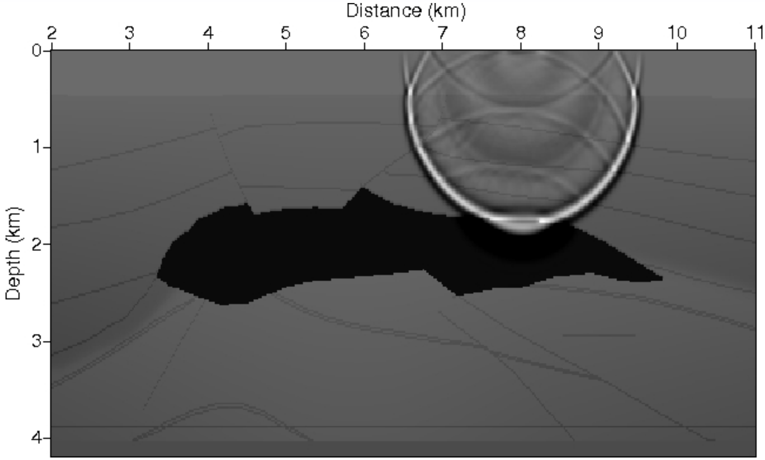
Back

Chebyshev polynomial Q_{800}



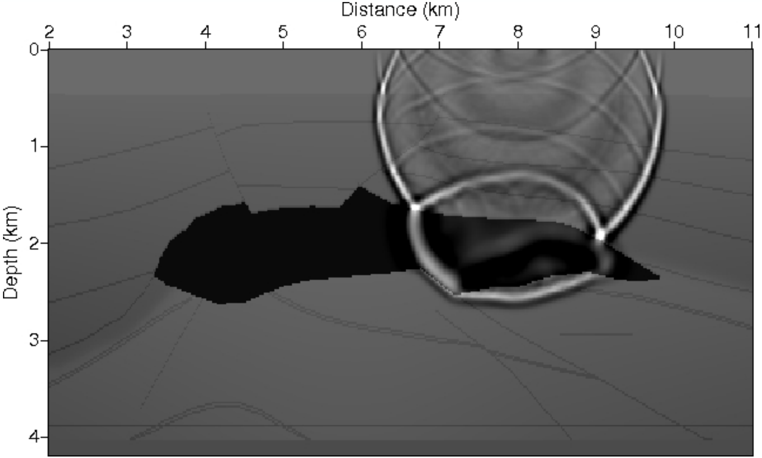
Back

Snapshot of the wavefield at 1.0 s



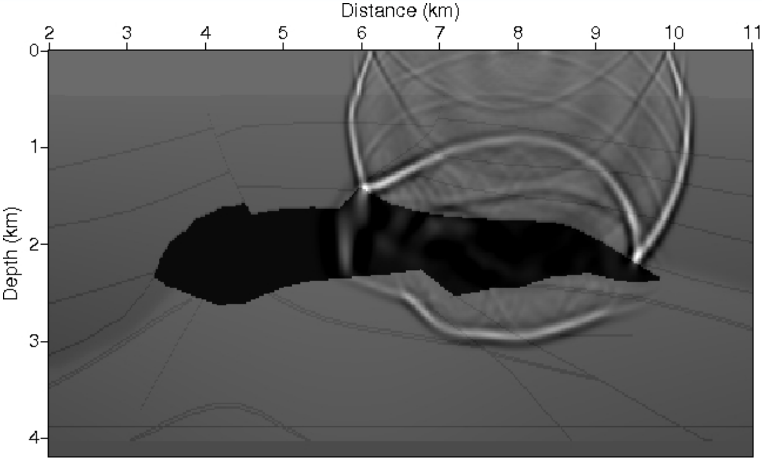
Back

Snapshot of the wavefield at 1.2 s



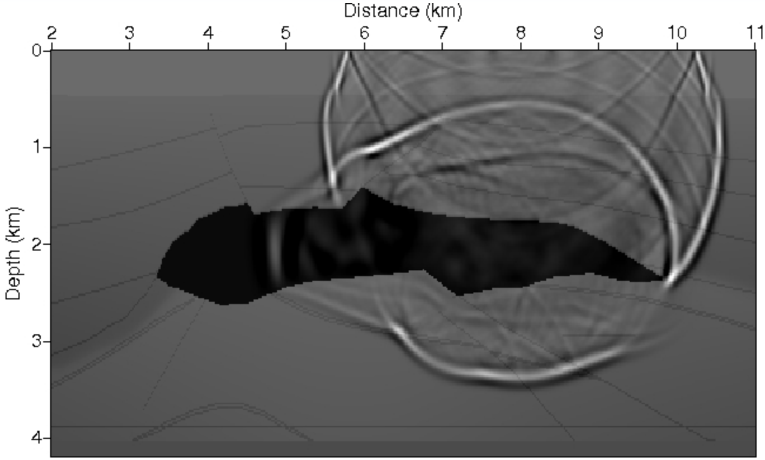
▸ Back

Snapshot of the wavefield at 1.4 s



Back

Snapshot of the wavefield at 1.6 s



▸ Back