Chebyshev expansion applied to the one-step wave extrapolation matrix

Reynam Pestana

CPGG/UFBA and INCT-GP/CNPQ
reynam@cpgg.ufba.br

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Acoustic wave equation

The acoustic wave equation in a source free medium with constant density is

$$\frac{\partial^2 p}{\partial t^2} = -L^2 p; \quad \text{with} \quad -L^2 = v^2 \nabla^2 \qquad (1)$$

where $\nabla^2 = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$ is the Laplacian operator,

$p = p(\mathbf{x}, t)$ is the pressure and $\mathbf{x} = (x, y, z)$ and

$v = v(\mathbf{x})$ is the compressional-wave velocity.

Equation (1) is a second order differential equation in the time variable.

Acoustic wave equation - An exact solution

Taking the wave equation (1)

$$\frac{\partial^2 p}{\partial t^2} = -L^2 p; \quad \text{with} \quad -L^2 = v^2 \nabla^2 \tag{2}$$

Initial conditions: $p(t=0) = p_0$ and $\frac{\partial p}{\partial t}(t=0) = \dot{p}_0$
Solution:

$$p(t) = \cos(L\,t)\,p_0 + \frac{sin(L\,t)}{L}\,\dot{p}_0 \tag{3}$$

The wavefields $p(t+\Delta t)$ and $p(t-\Delta t)$ can be evaluated by equation (3). Adding these two wavefields results in:

$$p(t+\Delta t) + p(t-\Delta t) = 2\cos(L\Delta t)\,p(t) \tag{4}$$

Standard finite-difference schemes

$$p(t+\Delta t) + p(t-\Delta t) = 2\cos(L\Delta t)\,p(t)$$

Taking the Taylor series expansion of $\cos(L\Delta t)$.

Second order: $(1 - \frac{(L\Delta t)^2}{2})$

$$p(t+\Delta t) - 2\,p(t) + p(t-\Delta t) = -\Delta t^2\,L^2\,p(t) \tag{5}$$

Fourth order: $(1 - \frac{(L\Delta t)^2}{2} + \frac{(L\Delta t)^4}{24})$

$$p(t+\Delta t) - 2\,p(t) + p(t-\Delta t) = -\Delta t^2\,L^2\,p(t) + \frac{\Delta t^4}{12}\,L^4\,p(t) \tag{6}$$

Standard finite-difference schemes (Etgen,1986; Soubaras and Zhang, 2008).

Explicit finite scheme

Using the 2nd order in time and higher-order finite differences, the forward propagation can be calculated as:

$$p_{i,j,k}^{n+1} = 2p_{i,j,k}^{n} - p_{i,j,k}^{n-1} + \Delta t^2 \, v_{i,j,k}^2 \left\{ (\nabla^2)^M \right\} p_{i,j,k}^{n} \qquad (7)$$

where,

$$p_{i,j,k}^{n} = p(i\Delta x, j\Delta y, k\Delta z, n\Delta t)$$

and Δt is temporal step size and Δx,Δy, Δz are spatial sampling interval.

Explicit finite scheme

The Laplacian with Mth order of accuracy can be given by

$$
\begin{aligned}
\left\{(\nabla^2)^M\right\} p^n_{i,j,k} &= w_o \left(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2} \right) p^n_{i,j,k} \\
&+ \sum_{m=1}^{M/2} w_m \left\{ \frac{1}{\Delta x^2} \left(p^n_{i-m,j,k} + p^n_{i+m,j,k} \right) \right. \\
&+ \frac{1}{\Delta y^2} \left(p^n_{i,j-m,k} + p^n_{i,j+m,k} \right) \\
&+ \left. \frac{1}{\Delta z^2} \left(p^n_{i,j,k-m} + p^n_{i,j,k+m} \right) \right\}
\end{aligned}
\tag{8}
$$

Stability condition

The stability condition for isotropic modeling is as follow (Lines et al. 1999)

$$\Delta t < \frac{\Delta d}{v_{max}} \frac{2}{\sqrt{\mu}} \tag{9}$$

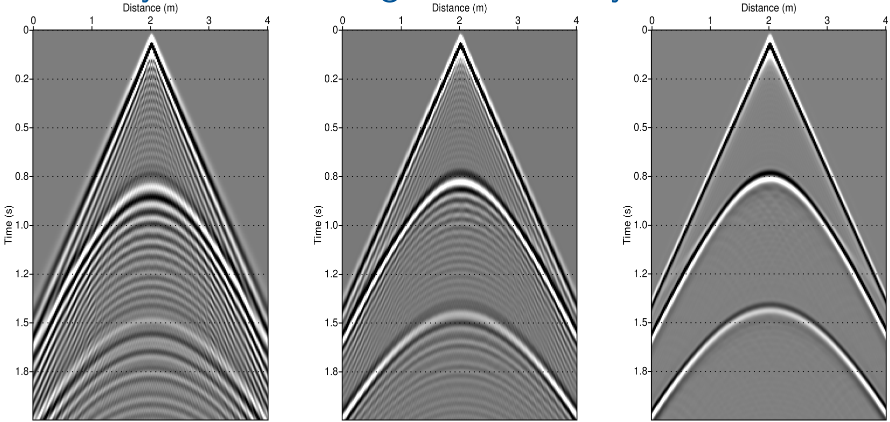
$$\mu = \sum_{m=-M/2}^{m=M/2} \left(|w_m^x| + |w_m^y| + |w_m^z| \right)$$

where $\Delta d = \min(\Delta x, \Delta y, \Delta z)$ and v_{max} is the maximum velocity in the medium.

The grid spacing is governed by maximum frequency or,

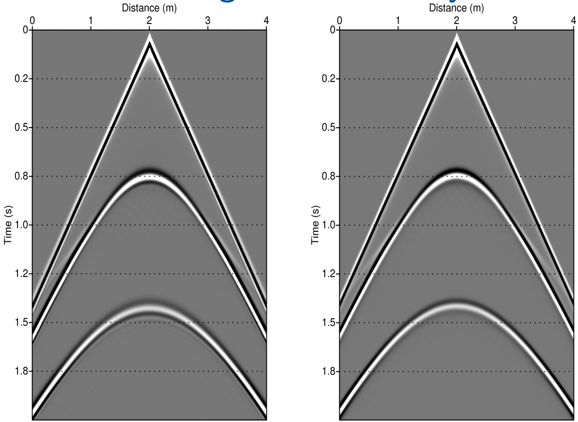
$$F_{max} = \frac{v_{min}}{G \Delta d}$$

Syntethic seismograms - Flat layers model



Modeled with different FD schemes with $F_{max} = 30Hz$, $\Delta d = 20m$ and 2.5 points per short wavelength. 2nd order (left), 4th order (center) and 14th order (right)

Synthetic seismograms - Flat layers model



Modeled with 2nd order (left) and 4th order (right) with $F_{max} = 30Hz$ and $\Delta d = 5.0m$.

Acoustic wave equation - An exact solution

$$\frac{\partial^2 p}{\partial t^2} = -L^2 p; \quad \text{with} \quad -L^2 = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) \tag{10}$$

Initial conditions:

$$p(t=0) = p_0 \qquad \frac{\partial p}{\partial t}(t=0) = \dot{p}_0$$

Solution:

$$p(t) = \cos(L\,t)\,p_0 + \frac{sin(L\,t)}{L}\,\dot{p}_0 \tag{11}$$

The wavefields $p(t+\Delta t)$ and $p(t-\Delta t)$ can be evaluated by equation (11). Adding these two wavefields results in:

$$p(t+\Delta t) + p(t-\Delta t) = 2\cos(L\Delta t)\,p(t) \tag{12}$$

The Rapid Expansion Method (REM)

The cosine function is given by (Kosloff et. al, 1989)

$$\cos(L\Delta t) = \sum_{k=0}^{M} C_{2k}\, J_{2k}(R\,\Delta t)\, Q_{2k}\left(\frac{iL}{R}\right) \tag{13}$$

Chebyshev polynomials recursion is given by:

$$Q_{k+2}(w) = (4w^2 + 2)\, Q_k(w) \;-\; Q_{k-2}\,(w)$$

with the initial values: $Q_0(w) = 1$ and $Q_2(w) = 1 + 2w^2$

For 3D case: $R = \pi c_{max} \sqrt{\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2} + \frac{1}{\Delta z^2}}$,

The summation can be safely truncated with a $M > R\,\Delta t$ (Tal-Ezer, 1987).

Laplace evaluation: $-L^2 = v^2\nabla^2$

- Fourier transformation scheme :

$$\frac{\partial^2 p}{\partial x^2} = IFFT[-k_x^2\, FFT[p(x)]]$$

- Finite difference:

$$\frac{\partial^2 p_j^n}{\partial x^2} \approx \frac{\delta^2 p_j^n}{\delta x^2} = \frac{1}{\Delta x^2} \sum_{l=-N}^{N} C_l\, p_{j+l}^n$$

- Convolutional filter (FIR):

$$FIR(l) = D_2(l) * H(l)$$

where $D_2(l) * H(l)$ is a Hanning tapered version of the standard operator $D_2(l)$

Separable Approximation

Normally, operators used in seismic imaging can be approximated as a series of separable terms such as

$$2\cos[L(\mathbf{x},\mathbf{k})\,\Delta t] \approx \sum_{j=0}^{N} a_j(\mathbf{x})\, b_j(\mathbf{k}) \tag{14}$$

where n is the number of terms in the series. Thus, extrapolation in time is then approximated by

$$p(\mathbf{x}, t+\Delta t) + p(\mathbf{x}, t-\Delta t) \approx$$
$$\approx \sum_{j=0}^{n} a_j(\mathbf{x}) \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} b_j(\mathbf{k})\, P(\mathbf{k}, t)\, e^{i(\mathbf{k}\cdot\mathbf{x})}\, d\mathbf{k} \tag{15}$$

Separable Approximation

The time wave propagation can be performed in the following way:

$$p(\mathbf{x}, t + \Delta t) = - p(\mathbf{x}, t - \Delta t) + \\ + \sum_{j=1}^{N} a_j(v) \, FFT^{-1} \, b_j(\mathbf{k}) \, FFT \, p(\mathbf{x}, t) \qquad (16)$$

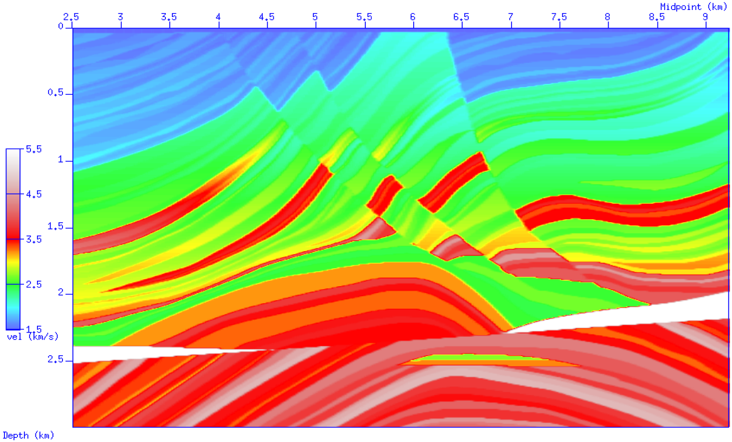
For the 2D case, each $b_j(\mathbf{k})$ is given by

$$b_j(\mathbf{k}) = \cos(v_j \sqrt{k_x^2 + k_z^2} \, \Delta t)$$

For each marching time step, this method requires one fast Fourier transform (FFT) and N inverse fast Fourier transforms (IFFT).

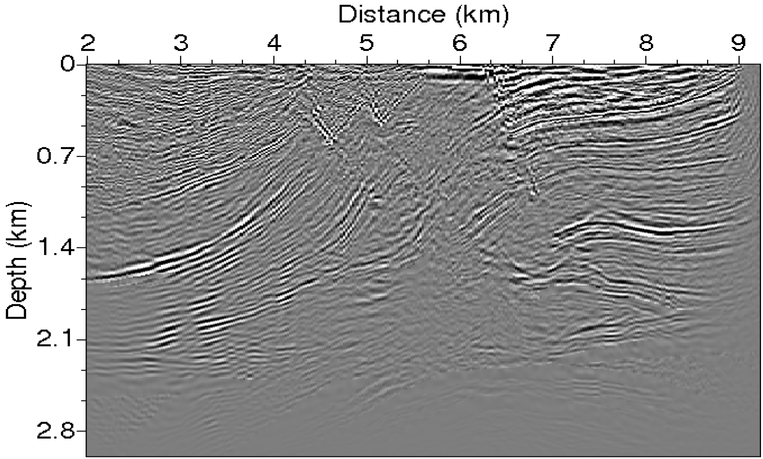
Marmousi Dataset

Velocity model



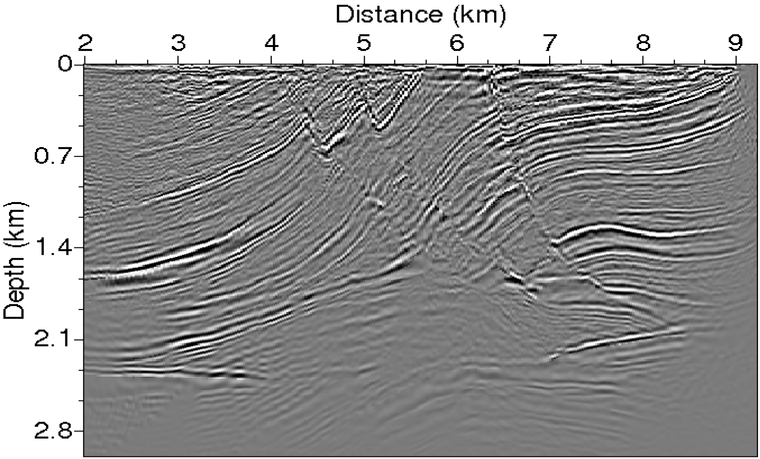
Marmousi Dataset

Reverse time migration using 3 velocities



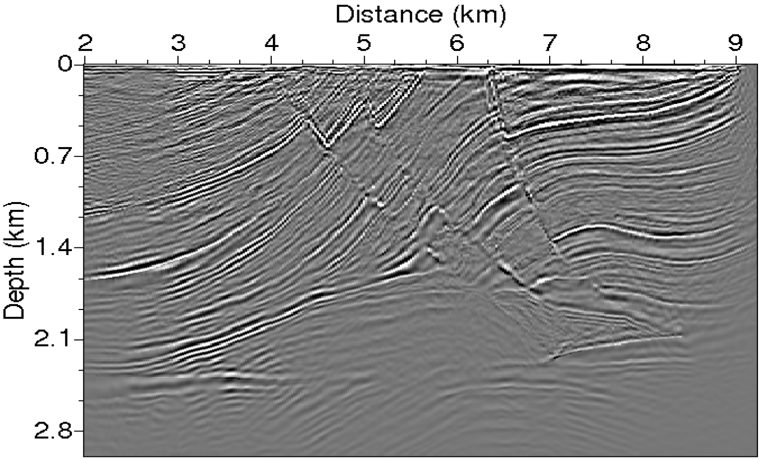
Marmousi Dataset

Reverse time migration using 5 velocities



Marmousi Dataset

Reverse time migration using 10 velocities



One-step wave extrapolation matrix

The pressure wavefield $\hat{P}$ is now a complex wavefield (Zhang and Zhang, 2009) defined as

$$\hat{P}(x,z,t) = p(x,z,t) + i\ q(x,z,t) \tag{17}$$

where $q(x,z,t) = H[p(x,z,t)]$.

For general media, the complex pressure wavefield $\hat{P}$ satisfies the following first-order partial equation in the time direction :

$$\frac{\partial \hat{P}}{\partial t} + i\Phi\hat{P} = 0. \tag{18}$$

where $\Phi = v\sqrt{-\nabla^2}$ and its symbol is $\phi = \ v(x,z)\ \sqrt{k_x^2 + k_z^2}$.

One-step wave extrapolation matrix

Considering a velocity constant case, i.e., $v = V$, the solution is:

$$\hat{P}(\mathbf{x}, t + \Delta t) = FFT^{-1} e^{-i\,V\,\sqrt{k_x^2 + k_z^2}\,\Delta t} FFT\ \hat{P}(\mathbf{x}, t). \qquad (19)$$

For variable velocity, the solution can be symbolically written as

$$\hat{P}(\mathbf{x}, t + \Delta t) = e^{-i\,\Phi\,\Delta t} \hat{P}(\mathbf{x}, t). \qquad (20)$$

where $\Phi = v\sqrt{-\nabla^2}$ and its symbol is $\phi = \ v(x, z)\,\sqrt{k_x^2 + k_z^2}$.

Numerical solution of the one-step wave extrapolation matrix

Zhang and Zhang (2009) applied a method based on an optimized separable approximation (OSA) which was proposed by Song (2001).

$$e^{-i\,\phi\Delta t} \approx \sum_{n=1}^{N} a_n(V)\, b_n(k)$$

$a_n(V)$ and $b_n(k)$ are the left and right eigenfunctions of the two dimension function $A(V,k) = exp(-iVk\Delta t)$.

where $V \in [V_{min}, V_{max}]$, $k = \sqrt{k_x^2 + k_z^2} \in [k_{min}, k_{max}]$.

For every time step extrapolation, the OSA method requires one fast Fourier transform (FFT) and N inverse fast Fourier.

First order wave equation in time - Chebyshev expansion

$$\frac{\partial \hat{P}}{\partial t} + i\Phi\hat{P} = 0. \tag{21}$$

$$\hat{P}(x,z,t) = p(x,z,t) + i\; q(x,z,t) \tag{22}$$

$$\frac{\partial U}{\partial t} = A\,U, \qquad \text{with} \qquad A = \begin{pmatrix} 0 & \Phi \\ -\Phi & 0 \end{pmatrix}, \tag{23}$$

where $U = [p,\; q]^T$ is the $2N_x \times N_z$ component vector of the pressure and Hilbert transform of the pressure wavefield and A is a matrix.

First order wave equation in time - Chebyshev expansion

The solution of the differential equation 23 is given by:

$$U(t+\Delta t) = e^{A\Delta t}\, U(t) \tag{24}$$

Now, to compute e^{At}, we start with the Jacobi-Anger approximation:

$$e^{ikRcos\theta} = \sum_{n=0}^{M} \varepsilon_n \; i^n J_n(kR)\, \cos(n\theta) \tag{25}$$

where $\varepsilon_0 = 1, \varepsilon_n = 2, \; n \geq 1$ and J_n represents the Bessel function of order n.

First order wave equation in time - Chebyshev expansion

For $z = i\cos\theta$, we have that $Q_n(z) = i^n \cos(n\theta)$

The modified polynomials of Chebyshev and they satisfy the following recurrence relation:

$$Q_{n+1}(z) = 2z\, Q_n(z) + Q_{n-1}(z); \tag{26}$$

with $\quad Q_0(z) = I \quad$ and $\quad Q_1(z) = z.$

Choosing $k = \Delta t$ and $z = A/R$, we obtain:

$$e^{A\Delta t} = \sum_{n=0}^{M} \varepsilon_n\, J_n(\Delta t R)\, Q_n(A/R) \tag{27}$$

First order wave equation in time - Chebyshev expansion

The first question about this expansion is the value of R.

The matrix A is anti-symmetric ($A^T = -A$) - the eigenvalues are all pure imaginary.

To guarantee that θ is real, $R = |\lambda_{max}|$ i.e, R has to be the maximum eigenvalue of A.

Here, we have for the 2D case, that

$$R = v_{max}\pi\sqrt{(1/\Delta x)^2 + (1/\Delta z)^2}$$

To guarantee that the series converges, we must assure that $M > R\Delta t$ (Tal-Ezer, 1986)

First order wave equation in time - Chebyshev expansion

Summarizing, the scheme using the Tal-Ezer's technique is written as:

$$\begin{pmatrix} p(t+\Delta t) \\ q(t+\Delta t) \end{pmatrix} = \sum_{n=0}^{M} \varepsilon_n \, J_n(\Delta t R) \, Q_n \left\{ \frac{A}{R} \right\} \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} \tag{28}$$

Now, we can compute all Chebyshev polynomial terms using its recurrence relation that is now given by:

$$\begin{aligned} Q_{n+1} \left[\frac{A}{R} \right] \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} &= 2 \left(\frac{A}{R} \right) \; Q_n \; \left[\frac{A}{R} \right] \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} \\ &+ \; Q_{n-1} \; \left[\frac{A}{R} \right] \begin{pmatrix} p(t) \\ p(t) \end{pmatrix} \end{aligned} \tag{29}$$

First order wave equation in time - Chebyshev expansion

where:

$$Q_0\left[\frac{A}{R}\right]\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \begin{pmatrix} p(t) \\ q(t) \end{pmatrix} \tag{30}$$

and

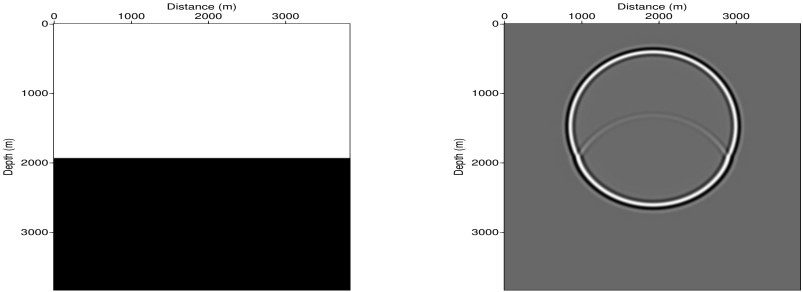
$$Q_1\left[\frac{A}{R}\right]\begin{pmatrix} p(t) \\ q(t) \end{pmatrix} = \frac{1}{R}\begin{pmatrix} \Phi\ p(t) \\ -\Phi\ q(t) \end{pmatrix} \tag{31}$$

To numerically implement this system, we need to compute the Φ operator applied on both p and q wavefields.

$$\Phi[p] = v(x,z)\ FFT^{-1}\left[\sqrt{k_x^2 + k_z^2}\ \ FFT(p)\right] \tag{32}$$

Numerical Examples

The two-layer model - PS method with $\Delta t = 1.0$ ms

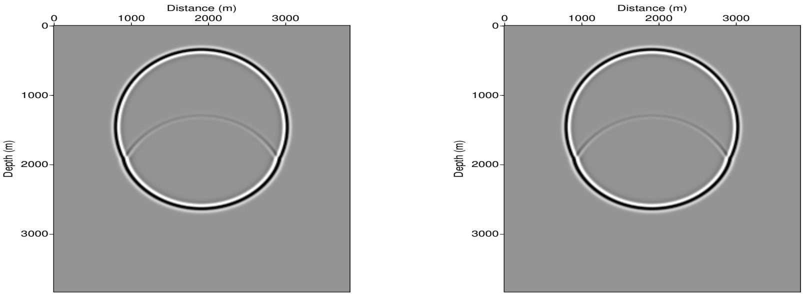


(a) (b)

Two layel velocity model (a) ; Snapshot at 0.6 s computed by the conventional pseudospectral method using a time-step value of 1.0 ms (b); Source location: x_s =1920 m, z_s = 1470 m

Numerical Examples

The two-layer model - ($F_{max} = 50$ Hz; $\Delta t_{Nqy} = 10$ ms)

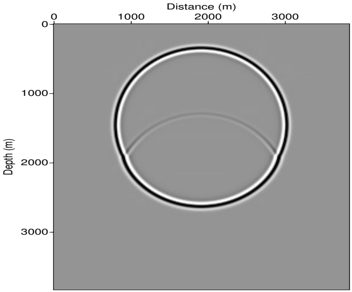


(a) (b)

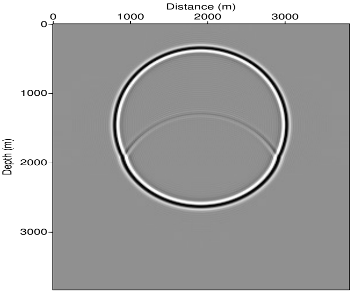
Chebyshev expansion method for the one-step wave extrapolation matrix using time-step values of 1.0 ms (a) and 4.0 ms (b).

Numerical Examples

The two-layer model



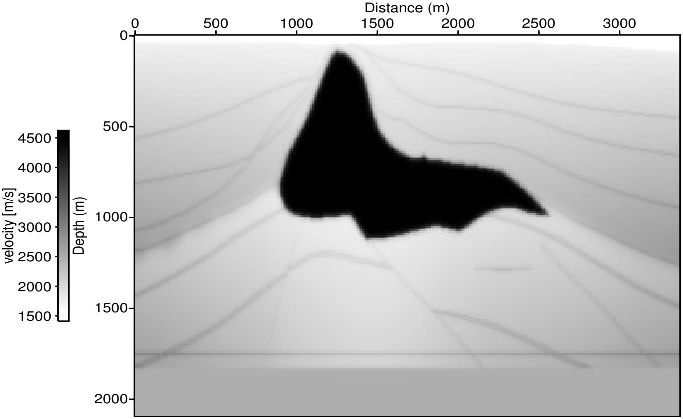
(a)



(b)

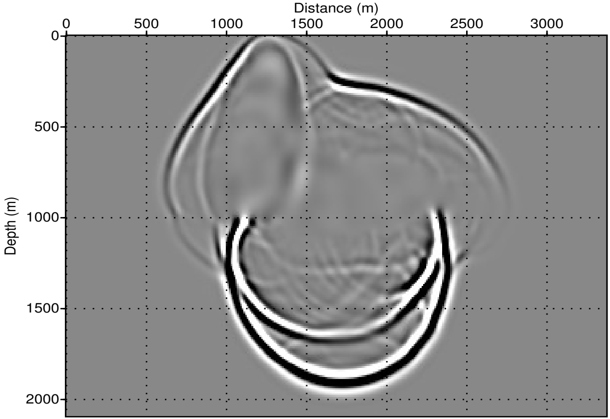
Chebyshev expansion method for the one-step wave extrapolation matrix using time-step values of 8.0 ms (a) and 10.0 ms (b).

Salt model: P-wave velocity model



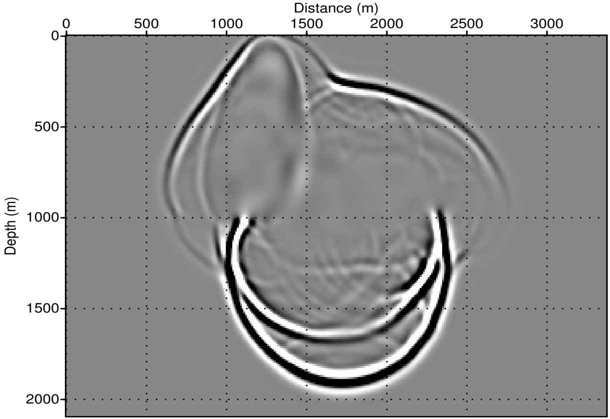
Source location: x_s =1690 m, z_s = 1200 m

Modeling using time-step value of 4.0 ms



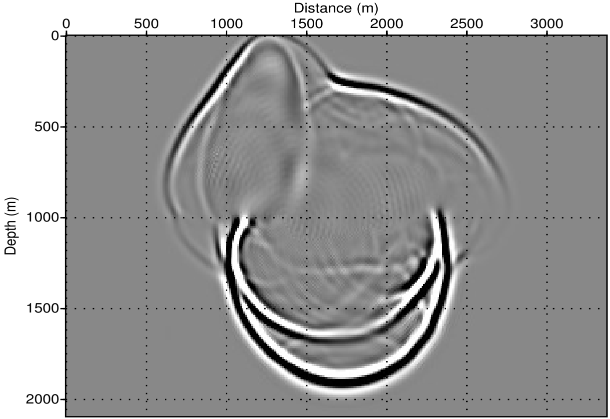
$F_{max} = 50$ Hz; $\Delta t_{nqy} = 10$ ms

Modeling using time-step value of 8.0 ms



$$F_{max} = 50 \text{ Hz}; \ \Delta t_{Nqy} = 10 \text{ ms}$$

Modeling using time-step value of 10.0 ms



$$F_{max} = 50 \text{ Hz}; \ \Delta t_{Nqy} = 10 \text{ ms}$$

Conclusions:

- The proposed numerical algorithm is based on the series expansion using the modified Chebyshev polynomials and with the pseudodifferential operator Φ computed using the Fourier method and the **proposed algorithm can handle any velocity variation.**
- The results demostrated that **the method is capable to extrapolate wavefieds in time up to the Nyquist time limit in a stable way and free of dispersion noise** when the number of terms of the Chebyshev expansion is appropriately chosen.
- Our method **can be easily extended to 3D problems** and can be applied for performing high quality modeling and imaging of seismic data.

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Thank you for your attention.