RAY THEORY AND GAUSSIAN BEAM METHOD FOR GEOPHYSICISTS

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RAY THEORY AND GAUSSIAN BEAM METHOD FOR GEOPHYSICISTS

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Apresentação

Desde a sua criação, a PETROBRAS tem investido recursos consideráveis na formação e desenvolvimento de seus empregados, através de treinamento interno e externo, incluindo iniciativas em parcerias com Universidades brasileiras. Através do seu Programa de Editoração de Livros Didáticos, a Universidade Petrobras se associa à Universidade Federal da Bahia para possibilitar a edição deste livro. A Empresa expressa assim apoios decisivos ao desenvolvimento científico. A idéia para a publicação deste livro surgiu em 1996, quando o Dr. Popov atuava como Professor no Curso de Pós-Graduação em Geofísica do Instituto de Geociências da Universidade Federal da Bahia. O livro representa uma importante contribuição para o estudo do chamado método dos feixes gaussianos, onde o autor é um renomado pesquisador. Este método tem importantes aplicações em geofísica de exploração, particularmente no processamento dos dados de reflexão sísmica, essenciais para a investigação de jazidas de petróleo.

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Foreword

Since its foundation in 1954, Petrobras has invested large amount of resources in its employees' education, both through internal/external training and partnerships with Brazilian Universities. Through its Books Publishing Program, the Petrobras University joins the Universidade Federal da Bahia–Brazil, to support the publication of this book. The Company thus establishes a positive support to scientific development.

The idea to publish this book arose in 1996, when Dr. Popov was working as a Professor in the Graduated Course in Geophysics at Universidade Federal da Bahia. This book plays a significant contribution for the study and applications of the so-called method of Gaussian beams, in which the author is a renowned researcher. That method is widely applied in exploration geophysics, particularly in the processing of reflection seismic data, which are of prime importance to the exploration of oil and gas accumulations.

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Preface

This book is based on a course of lectures delivered to postgraduate students in Geophysics at the Federal University of Bahia, Salvador, Brazil (CPGG, UFBa). In 1997-1998 these lectures were delivered at the Karlsruhe University, Germany. It is devoted to high-frequency approximation methods of the theory of wave propagation widely used in geophysical exploration, vz, the ray and the Gaussian beam methods. My main goal is to offer a consistent and reasonably detailed description of these methods which could provide their deeper understanding, enabling the reader to use them for solving geophysical exploration problems. Along with the classical results known in the ray theory since 1950s recent investigations on validity of the ray method are also included. As for the Gaussian beam method suggested in 1980s, I do not know of any monograph so far, which provides its consistent description.

I address this book to students and young scientists not very experienced in the field of high frequency propagation. That predetermines the methodological approach to the material of the book, which should be regarded as a textbook. I tried to follow some basic methodological requirements. Firstly, all material was put into strict order from simple to more complicated topics, and this order contains and preserves the main thread of the subject. Secondly, there should be no gaps in the mathematics underlying the topics, and detailed descriptions of basic technical tricks were consistently given.

For this reason the ray theory is described first for the reduced wave equation and then for the elastodynamic equations. The leading thread in the case of the ray theory which I accept and follow is to consider the ray method as an extension to inhomogeneous media and curved interfaces, with smoothly and slowly varying properties on the wavelength scale, of the theory of plane waves in homogeneous media with plane interfaces. I hope and believe that such a look about the ray theory should help the reader to build 'common sense' and a clearer understanding of the theory, in particular when reflection and transmission are discussed.

The Gaussian beam method stands aside this thread and is actually a more specific extension of the ray theory. It has the main advantage of enabling one to overcome caustic problems. However, the paraxial ray method acts as a bridge to the Gaussian beams, especially if we take into account the mathematical technique underlying the Gaussian beam method with, perhaps, one exception. A problem of decomposition of an initial wave field into Gaussian beams (or a problem of initial amplitudes for Gaussian beams) requires knowledge of asymptotic calculations of integrals of oscillating functions. At this point, in order to help the reader the basic ideas of the stationary phase method are described in the Appendix.

It becomes clear at first sight that both the ray and the Gaussian beam methods require at least some knowledge of advanced mathematics. I think that the only way to provide a deep understanding of the methods is simply to present the required mathematics in detail. So the reader will find auxiliary information on mathematics and most calculations are given in detail . It should be noticed that I do not use very modern and sometimes formal mathematical language, and try to simplify it as much as possible in accordance with the traditions of St. Petersburg Mathematical School. Therefore a refined specialist, perhaps, will be unsatisfied and will even find it outdated.

Obviously, the material in this book should be regarded as a tool for geophysical applications. Therefore, I try to illuminate the algorithmic essence of the methods under consideration and to supply examples where corresponding calculations are carried out analytically. They should also help the reader to master the subject.

Being conceived as a textbook, it differs from other books dealing with the ray method. In some of them numerous applications of the ray theory are described, while the mathematics underlying the theory is omitted. In others, the stress is made on modern mathematical language and technique, what seems to be rather complicated for a first study of the ray theory. Thus, I hope this book will suit a first detailed study of both the ray and the Gaussian beam methods.

It may be mentioned that the idea to present my lectures in the form of a book germinated during my second visit to CPGG in 1996. Being instrumental for my two previous visits, Prof. E. Sampaio acted as an editor helping me very much in preparing the book, and provided the opportunity and the facilities for its publication. I express my deep gratitude to him for that. Worth of mention are the contributions of Prof. O. Lima and Prof. M. Porsani for my present visit, which enabled the completion of the work. I am also kindly indebted to them.

I am also thankful to Mr. J. Lago for his hard and tedious work of typing the text and drawing the figures, and to Prof. H. Sato for improving the final version of the book. My former students have corrected several mistakes in the formulas and I am thankful to all of them, especially to Sergio Oliveira and Julian Celedon. I express my deepest gratitude to everyone in the staff of CPGG/UFBA, where I had the opportunity to work in a warm and friendly atmosphere. I also cordially thank Dr. I. Mufti for his valuable recommendations which helped in improving the presentation of the material. This work has been supported largely by the Brazilian Council of Scientific and Technical Development (CNPq) and partly by PETROBRAS.

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Introduction

Undoubtedly, the ray method remains one of the most powerful and broadly used methods for investigating both forward and inverse wave propagation problems in modern exploration geophysics. Some fundamental ideas of the ray theory have been known in physics for a long time. Probably, the first consistent description of the ray method, as a mathematical tool for studying wave propagation problems in electrodynamics, was given by R.K. Luneburg in his lectures in the mid 1940s. Unfortunately, his Lecture Notes were published much later - see Luneburg (1966), and, in fact, they remain inaccessible in scientific libraries.

In 1956, V.M. Babich suggested the ray method for the elastodynamic equations in the case of isotropic inhomogeneous media. Though this three-paged-article was published in a famous Russian journal, it has never been translated into English.

In 1959, F.C. Karal and J.B. Keller, succeeding Luneburg's Lecture Notes, published a consistent description of the ray method in isotropic elastodynamics. Thus, starting from the late 1950s the ray method begins its triumphal path in geophysics. Many important aspects of the ray theory and its applications can be found in monographs by Červený and Ravindra (1971), Červený, Molotkov and Pšenčik (1977), Hunyga (1984), Kravtsov and Orlov (1990).

The ray method has two main advantages. It provides a physical insight to the wave propagation phenomena in rather complicated geophysical models, by describing the total wave field as a sum of different types of waves generated in the problem under consideration. It gives rise to rather effective and not so time consuming numerical algorithms when compared to the finite difference and finite element methods. On the other hand, the ray method suffers the so-called caustic problems, which means precisely that if the rays associated with the wave propagation problem under investigation touch a surface in 3D (or a curve in 2D), the geometrical spreading vanishes and the ray amplitude becomes singular on it, while the real wave field remains finite and smooth. This surface is called the envelope to the ray field, or the caustic surface. Therefore, the ray method does not correctly describe the wave field in the vicinity of the caustics. Unfortunately, in geophysical models of elastic media the behavior of rays is complicated and they normally form many caustics of different geometrical structures. Thus, the ray method faces serious and unpleasant caustic problems in geophysical applications.

In 1965, V.P. Maslov suggested a new method, which provides a uniform mathematical representation for high frequency asymptotics of the wave field everywhere including caustics of arbitrary geometrical structures. This method is known now as Maslov's method, or the method of canonical operator - see Maslov (1965) and Maslov and Fedoryk (1976). Being elegant from the mathematical point of view and efficient in pure theoretical investigations, it leads to a numerical algorithm that strongly depends upon the geometrical structure of caustics formed by the rays, and therefore can hardly be regarded as a technological one.

On the other hand, the Gaussian beam method gives rise to a numerical algorithm for the calculation of the high frequency wave field independently of the position of an observation point. Briefly, it can be explained as follows. To calculate the wave field at an observation point M, we have to cover the vicinity of M by a fan of rays distributed in this vicinity more or less uniformly. For each ray we have to derive a Gaussian beam propagating along the ray and then to sum the contribution of each Gaussian beam to the point M over all rays from the fan. As every Gaussian beam has no singularity on caustics, the numerical algorithm does not depend on the position of M with respect to a caustic and on the geometrical structure of the caustic. The Gaussian beam method, as a new approach for the computation of wave fields in high frequency approximation, was suggested by M.M. Popov in his theoretical papers - see Popov (1981, 1982). The results of the first numerical experiments were presented in works by Kachalov and Popov (1981) and Červený, Popov and Pšenčik (1982). The survey article by Babich and Popov (1989) contains the review of the main ideas and numerical results concerning the Gaussian beam method and its relation with other asymptotic methods.

This book contains a consistent description of the ray theory and the Gaussian beam method for inhomogeneous isotropic elastic media with smooth interfaces, and includes also the paraxial ray theory which, in fact, makes a bridge between them. The material of the book is supposed to be presented in a self-sufficient form in such a way that the reader could master it without constantly looking at manuals on mathematics. Auxiliary mathematical details, necessary for the understanding, are included into the main text.

In Chapter 1 the main ideas of the ray theory and its mathematical technique are demonstrated on, perhaps, the simplest example of the scalar wave equation when they are not hidden under heavy mathematics. It also includes a relatively new approach to the problem of validity of the ray method, based on the asymptotic character of the ray series. This approach was suggested and examined on some model problems by Popov and Camerlynck (1996).

Chapter 2 contains a complete theory of the eikonal equation based on the variational calculus. This theory underlies the ray method both for the acoustic wave equation and for the elastodynamic equations. Auxiliary material on variational calculus are included in the text.

Chapter 3 is devoted to the theory of transport equation. Though the theory is not universal and requires additional development in the case of elastodynamic equations, it contains the basic concept of the ray method, namely the ray coordinates and geometrical spreading. The chapter is important for understanding the ray theory in elastodymanics presented in Chapter 7.

The essence of Chapter 4 is to illustrate the physical statement that the energy of the wave field in ray approximation propagates along ray tubes. It also contains a discussion on the caustic problem, which turns out to be the main obstacle for the applications of the ray method.

The main idea of the paraxial ray theory presented in Chapter 5 can be formulated as follows. If we consider a narrow ray tube surrounding a fixed central ray of the tube, we may linearize Euler's equations for the rays from the tube because they are close to the central one. This results in a significant simplification of Euler's equations, and therefore of the whole ray theory built on them. These equations are known in classical mechanics as equations in variations. Of course, that theory remains valid only in the vicinity of the central ray. In fact, the paraxial ray theory was used and developed by specialists in optical resonators for lasers in the 1960s. In particular, the ray centered coordinates, broadly used in the paraxial theory and in the Gaussian beam method, were introduced by Popov (1969) in studies on self oscillations in opened resonators. In geophysics the paraxial approach was firstly used by Popov and Pšenčik (1978), see also Popov (1977), for the dynamic ray tracing problem. The chapter includes the solution of the latter problem presented in a slightly different way from the original paper. Reduction to 2D and to 2.5D and the example of constant gradient velocity model accompanies the main material in order to facilitate understanding.

Chapter 6 starts with a preparatory material for the successive chapter devoted to the ray method in elastodynamics. It also contains the solution of the initial problem for geometrical spreading on a smooth interface, which completes the dynamic ray tracing problem from the previous chapter.

Chapter 7 constitutes the main body of the book, and concerns the ray theory in elastodynamics. Started with the auxiliary topic of plane waves in a homogeneous elastic medium, it contains the derivation of the eikonal and transport equations, and the solution of the initial data problem for the ray amplitudes in the case of point sources for general isotropic inhomogeneous elastic medium. In the last section, the validity problem of the ray theory is discussed. A simple example shows that the second term of a ray series may increase with an increase in distance between a source and an observation point, thus leading to a strong limitation in the use of the ray method.

Chapter 8 is devoted to the theory of the Gaussian Beam method and its applications to direct wave propagation problems. Individual gaussian beams as asymptotic solutions to Maxwell's equations and Helmholtz's wave equation were developed by specialists in optical resonators for gas lasers in the 1960s. As for the elastodynamic equations, N.Ya. Kirpichnikova (1971) developed Gaussian beams in inhomogeneous isotropic media. The integral over Gaussian beams as an asymptotic representation of a scalar wave field in the high frequency approximation was firstly suggested by Babich and Pankratova (1972) in pure mathematical studies. The consistent description of the Gaussian Beam method in elastodynamics was given by Popov (1983). The material of the previous two chapters allows for significant simplification of the theory of an individual Gaussian beam. This theory is illustrated by a simple example and by a reduction to 2D case. The results of numerical experiments are taken from papers by Kachalov and Popov (1985,1988). More applications of the Gaussian Beam method can be found in the review papers by Červený (1985) and by Babich and Popov (1989).

The main aim of the conclusive Chapter 9 is to discuss the relationship between the asymptotics of the wave field in the frequency and time domain and to point out some of its peculiarities. It has been known for a long time that the Fourier transform is the proper tool for making a bridge between them. However, there is another approach based on the use of the so-called space-time Gaussian beams. This approach, suggested by Popov (1987), is briefly discussed in the chapter. But the full theory turns out to be more complicated and therefore is not presented in the book.

In the Appendix, the main ideas of the stationary phase method are illustrated by an example of a single integral having one non-degenerate critical point. For more a detailed description of the method and complicated cases of manifold integrals see Smirnov (1964), Bleistein and Handleman (1986) and Wong (1989).

1

Basic equations of the ray theory for the reduced wave equation

1.1 Main ideas leading to the ray theory

Consider the wave equation

$$\Delta W - \frac{1}{C^2}\frac{\partial^2 W}{\partial t^2} = 0\,,$$

where t means time,

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \quad \text{is the the Laplace operator,}$$

C is the velocity of propagation of the wave field W. Suppose that C is constant.

Let us seek a solution in the following form

$$W = e^{i\varphi} = \cos\varphi + i\sin\varphi, \quad i^2 = -1, \quad \varphi = -\omega t + k_1 x + k_2 y + k_3 z\,,$$

where ω, k_1, k_2, k_3 are constants.

In this case we have

$$\frac{\partial W}{\partial t} = e^{i\varphi} i \frac{\partial \varphi}{\partial t} = -i\omega W\,; \quad \frac{\partial^2 W}{\partial t^2} = (-i\omega)^2 W\,;$$

$$\frac{\partial W}{\partial x} = ik_1 W\,; \qquad \frac{\partial^2 W}{\partial x^2} = (ik_1)^2 W\,.$$

Therefore

$$\Delta W - \frac{1}{C^2}\frac{\partial^2 W}{\partial t^2} = W\left[-\frac{(-i\omega)^2}{C^2} + (ik_1)^2 + (ik_2)^2 + (ik_3)^2\right] = 0 \quad .$$

Due to $W \not\equiv 0$ it necessarily means that

$$\frac{\omega^2}{C^2} = k_1^2 + k_2^2 + k_3^2 \,. \tag{1.1}$$

Definition: ω is called the circular frequency, $\vec{k} = k_1\vec{i} + k_2\vec{j} + k_3\vec{k'}$ is the wave vector and

$$|\vec{k}| = \sqrt{k_1^2 + k_2^2 + k_3^2} = \frac{2\pi}{\lambda}$$

where λ is the wavelength.

Now from (1.1) we get

$$\frac{\omega^2}{C^2} = |\vec{k}|^2 \to \frac{\omega}{C} = |\vec{k}| = \frac{2\pi}{\lambda} \,.$$

Thus we obtain a solution in the form of a plane wave

$$W = Ae^{i(-\omega t + k_1 x + k_2 y + k_3 z)}$$

where A is the constant amplitude of the plane wave, and $\varphi = -\omega t + k_1 x + k_2 y + k_3 z$ is the phase of the plane wave.

Why is it called plane wave?

Let us introduce the radius-vector $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k'}$, then

$$\varphi = -\omega t + (\vec{k}, \vec{r})$$

where, as usual, $(\vec{k}, \vec{r})$ means the scalar product between the vectors $\vec{k}$ and $\vec{r}$.

Consider a surface on which the phase is fixed (Fig. 1.1), say,

$$\varphi = 0 \to (\vec{k}, \vec{r}) = \omega t \,,$$

so for each moment t this is a plane in 3D with vector $\vec{k}$ being orthogonal to it.

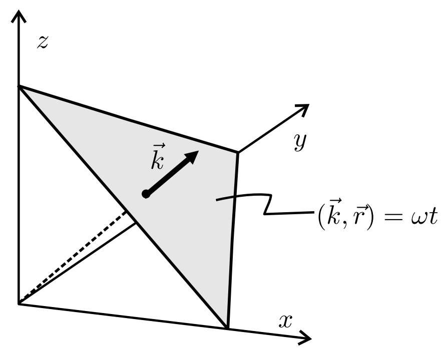

Figure 1.1: A surface of constant phase $\varphi = 0$ in a 3D case.

If we consider further $\vec{r} = (\vec{k}/|\vec{k}|)s$, then $(\vec{k}, \vec{r}) = \omega t$ takes the form $|\vec{k}|s = \omega t$. It follows now that

$$|\vec{k}|\frac{ds}{dt} = \omega \rightarrow \frac{ds}{dt} = \frac{\omega}{|k|} = C \,.$$

Thus, the surface of constant phase moves in space with a velocity C in the fixed direction by the vector $\vec{k}$.

Plane wave solutions play a remarkable role in mathematical physics because many types of solutions can be presented as a superposition of plane waves. Obviously, a plane wave solution does not exist if the velocity varies. But suppose that the velocity varies slowly. In this case it is natural to seek a solution for the wave equation in a form of the so-called deformed plane wave $W = A(t, x, y, z)e^{i\varphi(t,x,y,z)}$ where now the amplitude A is no longer constant, but depends on coordinates, and the phase function φ is not a linear function. We arrive at a special analytical form of an approximate solution of the wave equation which is called now the ray series, and we may assert that the ray method is an extension of the plane wave theory to slowly varying media.

1.2 Eikonal and transport equations; the problem of validity of the ray series

We shall consider further a wave field harmonic in time. It means that we assume

$$W(t, x, y, z) = e^{-i\omega t}U(x, y, z) \,.$$

By inserting the latter expression into the wave equation we get for U the reduced wave equation or Helmholtz equation

$$\left(\Delta + \frac{\omega^2}{C^2}\right) U = 0 \,.$$

Let us derive the eikonal and transport equations. Now $C = C(x, y, z)$ and we seek a solution in the form

$$U = e^{i\omega\tau(x,y,z)}A(x, y, z)$$

where τ is called eikonal, A is the amplitude and the circular frequency ω is supposed to be a large parameter. By differentiating U with respect to x we get

$$\frac{\partial U}{\partial x} = e^{i\omega\tau}\left(i\omega\frac{\partial \tau}{\partial x}A + \frac{\partial A}{\partial x}\right) \,,$$

$$\begin{aligned}
\frac{\partial^2 U}{\partial x^2} &= e^{i\omega\tau}\left\{i\omega\frac{\partial\tau}{\partial x}\left(i\omega\frac{\partial\tau}{\partial x}A + \frac{\partial A}{\partial x}\right) + i\omega\frac{\partial^2\tau}{\partial x^2}A + i\omega\frac{\partial\tau}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial^2 A}{\partial x^2}\right\} = \\
&= e^{i\omega\tau}\left\{-\omega^2\left(\frac{\partial\tau}{\partial x}\right)^2 A + i\omega\left(2\frac{\partial\tau}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial^2\tau}{\partial x^2}A\right) + \frac{\partial^2 A}{\partial x^2}\right\} \,.
\end{aligned}$$

Let us introduce additional notations

$$\begin{aligned} \text{grad } \tau &= \nabla\tau = \frac{\partial\tau}{\partial x}\vec{i} + \frac{\partial\tau}{\partial y}\vec{j} + \frac{\partial\tau}{\partial z}\vec{k'} , \\ (\text{grad } \tau, \text{grad } \tau) &= (\nabla\tau)^2 = \left(\frac{\partial\tau}{\partial x}\right)^2 + \left(\frac{\partial\tau}{\partial y}\right)^2 + \left(\frac{\partial\tau}{\partial z}\right)^2 , \\ (\text{grad } \tau, \text{grad } A) &= (\nabla\tau, \nabla A) = \frac{\partial\tau}{\partial x}\frac{\partial A}{\partial x} + \frac{\partial\tau}{\partial y}\frac{\partial A}{\partial y} + \frac{\partial\tau}{\partial z}\frac{\partial A}{\partial z} . \end{aligned}$$

By inserting $U = e^{i\omega\tau}A$ into the reduced wave equation and by taking into account previous formulas we obtain

$$\left(\Delta + \frac{\omega^2}{C^2}\right) U = e^{i\omega\tau}\left\{\omega^2\left(\frac{1}{C^2} - (\nabla\tau)^2\right) A + i\omega\big(2(\nabla\tau, \nabla A) + \Delta\tau A\big) + \Delta A\right\} .$$

We suppose ω is a large parameter of the problem and impose the following equations

$$(\nabla\tau)^2 = \frac{1}{C^2} \quad \text{(eikonal equation)} ;$$

$$2(\nabla\tau, \nabla A) + A\Delta\tau = 0 \quad \text{(transport equation)},$$

in order to eliminate larger terms. But ΔA remains and we have no chance to eliminate it. Thus, in this case we are not able to satisfy the equation exactly!

In order to decrease the discrepancy, we introduce an infinite series to U as follows

$$U \sim e^{i\omega\tau(x,y,z)} \sum_{n=0}^{\infty} \frac{A_n(x, y, z)}{(-i\omega)^n} \quad \text{(ray series)} .$$

By inserting this series into the reduced wave equation we obtain the eikonal equation

$$(\nabla\tau)^2 = \frac{1}{C^2}$$

and a recurrent set of transport equations

$$2(\nabla\tau,\ \nabla A_{n+1}) + A_{n+1}\Delta\tau = \Delta A_n , \quad n = -1, 0, 1 \cdots , \quad A_{-1} \equiv 0.$$

For $n = -1$ we get the transport equation for the main term of amplitude

$$2(\nabla\tau, \nabla A_0) + A_0\Delta\tau = 0 .$$

Let us dwell on the asymptotic character of the ray series and on the problem of validity of the ray method.

Definition of asymptotic series.

Consider a function $f(z)$ and let z be large, i.e. $z \to \infty$.

We say that $\sum_{n=0}^{\infty} a_n/z^n$ is an asymptotic series for the function $f(z)$ as $z \to \infty$ if for an arbitrary fixed N the following inequality holds true

$$\left| f(z) - \sum_{n=0}^{N} \frac{a_n}{z^n} \right| \le \frac{\text{const}}{z^{N+1}},$$

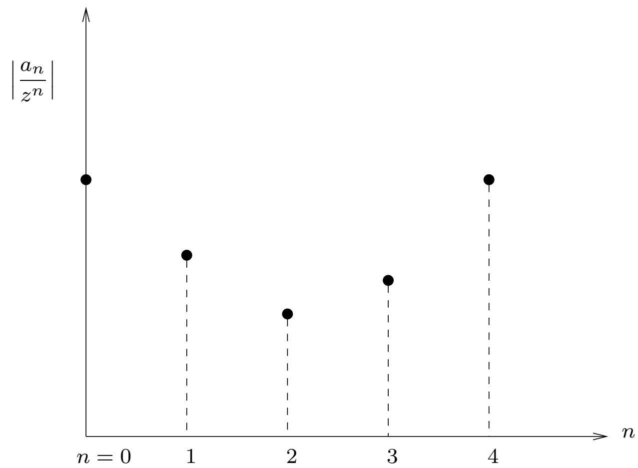

Figure 1.2: Typical behavior of terms in an asymptotic series as a function of number n.

where the const does not depend on z.

We write in this case

$$f(z) \sim \sum_{n=0}^{\infty} \frac{a_n}{z^n} .$$

Thus, by increasing z we can achieve on the right-hand side of the latter inequality a value as small as desired and therefore the approximation of $f(z)$ by the sum of N terms will be as precise as necessary. Suppose now that the value of z is restricted from above, which is exactly what we face in most applications. In this case the question of applicability of the asymptotics of $f(z)$ is rather different from the one discussed above. Indeed, a typical behavior of terms of an asymptotic series as a function of its number n is depicted in Fig. 1.2. There are some decreasing terms while all other terms usually increase. Of course, the position of the minimum on Fig. 1.2 depends upon the particular value of the argument z. In order to approximate the function $f(z)$ we have to take only the decreasing terms of the asymptotic series. If, for example, n_* is the number of the smallest term of the asymptotic series, then one cannot expect to achieve a better accuracy than $|a_{n_*}/z^{n_*}|$ in approximating $f(z)$ by the asymptotic series and, to this end, one has to take the sum of $(n_* - 1)$ terms. If however, there is no descending branch on Fig.1.2 for a given value of z the asymptotic series cannot be used for approximation $f(z)$ at all. Thus, the criteria of applicability of an asymptotic series should be linked to the existence of a descending branch of the asymptotic series.

Let us come back to the ray series.

Normally the ray series in the frequency domain does not converge and we may expect it to be an asymptotic series for a certain exact solution with respect to ω tending to infinity (Note: it is still a difficult mathematical problem to prove such a theorem!). If it is so, we may expect that, by using only the main term of the ray series, the following inequality holds

$$\left|U - e^{i\omega t} A_0\right| \leq \frac{\text{const}}{\omega} \, ,$$

where the const depends on geometrical parameters of the problem under investigation, i.e. the distance between a source and an observation point, curvatures of interfaces and so on (but not upon ω).

Unfortunately, it is very difficult to obtain an analytical expression for the const and therefore the latter inequality cannot be used for studying a region of validity of the ray method. By dealing with the ray series we are not able to study the behaviour of its coefficients as a function of the number, not even in simple model problems, because well-known formulas for the coefficients are given in implicit form and their analytical investigation seems to be an unrealistic problem.

Nevertheless, a quite reliable criterion of validity of the ray method can be given on the basis of two terms of the ray series.

If the ratio $|A_1/(\omega A_0)|$ is less than one for a given ω and other parameters of the problem under consideration, then we can approximate an exact solution U by the main term $A_0 e^{i\omega\tau}$ because in this case we will get decreasing terms of the ray series. But if this ratio is equal to one, the main term cannot be used for approximating U for a given frequency ω and other parameters at all.

Thus, to study a region of validity of the ray method on the basis of the criterion mentioned above we need to develop a computational algorithm for the second term of the ray method. We shall discuss this problem in more detail in Chapter 7.

2
Solution of the eikonal equation

2.1 Fermat's principle

Consider a 3D medium with a given velocity of wave propagation $C = C(x, y, z)$.

Suppose a signal or wave field propagates along a smooth curve l

$$l : \begin{cases} x = x(\sigma) \\ y = y(\sigma) \\ z = z(\sigma) \end{cases}$$

from a point A to a point B (Fig. 2.1). Let us compute the time necessary for the signal to travel from A to B along l.

Obviously, $dt = ds/C$ where ds is an element of length along l. By integrating over l we get

$$T[l] = \int_{(A)}^{(B)} \frac{ds}{C} \,.$$

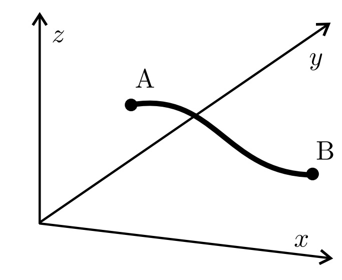

Figure 2.1: A curve l along which a signal propagates

This integral is called Fermat's integral or Fermat's functional.

It is a curvilinear integral, although it can be reduced to an ordinary integral. Indeed, we have

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{\dot{x}^2(\sigma) + \dot{y}^2(\sigma) + \dot{z}^2(\sigma)}d\sigma$$

where $\dot{x} = dx/d\sigma$. By taking into account the latter expression we get the ordinary integral

$$T[l] = \int_{\sigma_A}^{\sigma_B} \frac{\sqrt{\dot{x}^2(\sigma) + \dot{y}^2(\sigma) + \dot{z}^2(\sigma)}}{c\big(x(\sigma), y(\sigma), z(\sigma)\big)} d\sigma \equiv \int_{\sigma_A}^{\sigma_B} L(\dot{x}, \dot{y}, \dot{z}, x, y, z) d\sigma \ ,$$

where L is Lagrangian.

Fermat's principle: a signal (either light or a wave field) propagates along such a curve l between A and B that the integral T reaches a minimum on l.

Finding this curve l is a crucial issue in variational calculus.

2.2 Variation of a functional; Euler's equations

Consider the simplest case of Fermat's integral

$$T[l] = \int_{(A)}^{(B)} L(\dot{x}(\sigma), x(\sigma)) d\sigma \ .$$

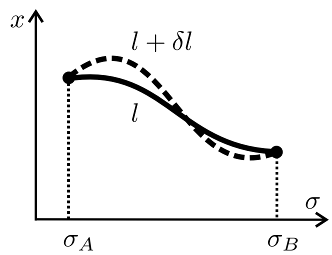

Figure 2.2: Variation of the curve l with fixed ends.

Let us compare T on different curves, i.e., consider curves $l + \delta l$ close to l (Fig. 2.2):

$$x = x(\sigma) + \delta x(\sigma)$$
$$\dot{x} = \dot{x}(\sigma) + \delta \dot{x}(\sigma)$$

where as usual $\bullet$ means $d/d\sigma$ and variation δx of the curve l is supposed to be small.

Let us develop $T[l+\delta l]$ as follows

$$\begin{aligned} T[l+\delta l] &= \int_A^B L(\dot{x}+\delta\dot{x}, x+\delta x)d\sigma = \\ &= \int_A^B L(\dot{x},x)d\sigma + \int_A^B \left\{ \frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial x}\delta x + O(\delta x)^2 \right\} d\sigma\ , \end{aligned}$$

where $O(\delta x)^2$ implies the terms of second and higher order with respect to the variations $\delta x, \delta\dot{x}$.

Further we get

$$\Delta T_{[l]} = T[l+\delta l] - T[l] = \int_A^B \left(\frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial x}\delta x \right) d\sigma + \int_A^B\ O(\delta x)^2 d\sigma\ ,$$

where the first integral describes the main linear part of the difference $\Delta T[l]$ with respect to small variations δx, $\delta\dot{x}$.

Definition: The linear part of the difference ΔT is called variation (or first variation) of the functional and is denoted by δT.

Thus, in our case

$$\delta T = \int_A^B \left(\frac{\partial L}{\partial \dot{x}}\delta\dot{x} + \frac{\partial L}{\partial x}\delta x \right) d\sigma\ .$$

Note $\delta\dot{x}$ and δx are not independent. By integrating in parts the first term under the integral, we get

$$\delta T = \frac{\partial L}{\partial \dot{x}}\delta x\Big|_A^B + \int_A^B \left(\frac{\partial L}{\partial x} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}} \right) \delta x d\sigma\ . \tag{2.1}$$

This is the formula for the first variation of the functional with mobile ends (it means that $\delta x\big|_A$ and $\delta x\big|_B$ may be arbitrary).

The first variation in the case of fixed ends reads

$$\delta T = \int_A^B \left(\frac{\partial L}{\partial x} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}} \right) \delta x d\sigma \tag{2.2}$$

because in this case

$$\delta x\big|_A = \delta x\big|_B = 0\ .$$

Theorem. If $T[l]$ gets min (max) on the curve l, then the first variation $\delta T[l]$ on l vanishes, i.e., $\delta T[l] = 0$.

Explanation : the proof of this theorem is very similar to the proof of a well-known theorem in classical analysis which deals with the necessary condition of max (min) at a point $x = x_o$ for a smooth function $f(x)$. Indeed, in the latter theorem we study the difference

$$\Delta f = f(x_0+\delta x) - f(x_0)$$

for a small δx and arrive at the conclusion that $df\big|_{x_o} = 0$ (or $\dfrac{df}{dx}\big|_{x_o} = 0$).

In case of a functional $T[l]$ the first variation $\delta T[l]$ is the main linear part of $\Delta T[l]$ (like differential df for Δf!) and changes its sign together with the sign of variation δx. But $\Delta T[l]$ preserves its sign in a vicinity of max (min), therefore $\delta T[l]$ must be equal to zero on the curve l.

Note: if $f'(x_0) = 0$ we may have max, min or flex point at $x = x_0$. For the functional we can also have these possibilities!

Consider the problem with fixed ends. Suppose $T[l]$ attains a minimum on the curve l, then

$$\delta T = \int_A^B \left(\frac{\partial L}{\partial x} - \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{x}} \right) \delta x d\sigma = 0$$

for an arbitrary δx (but small). It follows from here that on the curve l the following differential equation must hold

$$\frac{d}{d\sigma} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0.$$

This is called Euler's equation (it is an ordinary differential equation of second order with respect to function $x = x(\sigma)$).

Extension to a 3D case.

Let us consider the previous problem in 3D. Now the functional reads

$$T[l] = \int_A^B L(\dot{x}, \dot{y}, \dot{z}, x, y, z) d\sigma\ .$$

We introduce variations through the formulas

$$\begin{aligned} &x \to x + \delta x, \quad y \to y + \delta y, \quad z \to z + \delta z\ , \\ &\dot{x} \to \dot{x} + \delta \dot{x}, \quad \dot{y} \to \dot{y} + \delta \dot{y}, \quad \dot{z} \to \dot{z} + \delta \dot{z}\ , \end{aligned}$$

and by almost repeating the previous calculations we arrive at the following expression for the first variation of the functional

$$\begin{aligned} \delta T[l] = \int_A^B &\left\{ \left(\frac{\partial L}{\partial \dot{x}} \delta \dot{x} + \frac{\partial L}{\partial x} \delta x \right) + \left(\frac{\partial L}{\partial \dot{y}} \delta \dot{y} + \frac{\partial L}{\partial y} \delta y \right) + \left(\frac{\partial L}{\partial \dot{z}} \delta \dot{z} + \frac{\partial L}{\partial z} \delta z \right) \right\} d\sigma = \\ = &\left(\frac{\partial L}{\partial \dot{x}} \delta x + \frac{\partial L}{\partial \dot{y}} \delta y + \frac{\partial L}{\partial \dot{z}} \delta z \right) \Bigg|_A^B + \int_A^B \left\{ \left(\frac{\partial L}{\partial x} - \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{x}} \right) \delta x + \right. \\ &\left. + \left(\frac{\partial L}{\partial y} - \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{y}} \right) \delta y + \left(\frac{\partial L}{\partial z} - \frac{d}{d\sigma} \frac{\partial L}{\partial \dot{z}} \right) \delta z \right\} d\sigma\ . \end{aligned}$$

To derive Euler's equations we have to apply the previous theorem to $\delta T[l]$ and to take into account that all variations δx, δy, δz are independent.

Euler's equations for the case of $L = \frac{1}{C}\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}$ take the form

$$\begin{cases} \dfrac{d}{d\sigma}\dfrac{\partial L}{\partial \dot{x}} - \dfrac{\partial L}{\partial x} = 0 \to \dfrac{d}{d\sigma}\left(\dfrac{\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\dfrac{1}{C}\right) - \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\frac{\partial}{\partial x}\dfrac{1}{C} = 0 \\ \dfrac{d}{d\sigma}\dfrac{\partial L}{\partial \dot{y}} - \dfrac{\partial L}{\partial y} = 0 \to \dfrac{d}{d\sigma}\left(\dfrac{\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\dfrac{1}{C}\right) - \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\dfrac{\partial}{\partial y}\dfrac{1}{C} = 0 \\ \dfrac{d}{d\sigma}\dfrac{\partial L}{\partial \dot{z}} - \dfrac{\partial L}{\partial z} = 0 \to \dfrac{d}{d\sigma}\left(\dfrac{\dot{z}}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}}\dfrac{1}{C}\right) - \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}\dfrac{\partial}{\partial z}\dfrac{1}{C} = 0 \end{cases}$$

where σ is a parameter along a curve $x = x(\sigma)$, $y = y(\sigma)$, $z = z(\sigma)$.

Definition: Any solution of Euler's equations is called an extremal in variational calculus and a ray in geophysics.

Example 1. Let us take an arc length s along a ray instead of the arbitrary parameter σ, then

$$ds = \sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}d\sigma \to \frac{d}{ds} = \frac{d}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}d\sigma}$$

and we obtain

$$\frac{d}{ds}\left(\frac{1}{C}\frac{dx}{ds}\right) - \frac{\partial}{\partial x}\frac{1}{C} = 0\,,$$
$$\frac{d}{ds}\left(\frac{1}{C}\frac{dy}{ds}\right) - \frac{\partial}{\partial y}\frac{1}{C} = 0\,,$$
$$\frac{d}{ds}\left(\frac{1}{C}\frac{dz}{ds}\right) - \frac{\partial}{\partial z}\frac{1}{C} = 0\,,$$

or in vectorial form

$$\frac{d}{ds}\left(\frac{\vec{t}}{C}\right) - \text{ grad }\frac{1}{C} = 0\,,$$

where

$$\vec{t} = \frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j} + \frac{dz}{ds}\vec{k'}$$

is a unit vector tangent to a ray.

Example 2. Homogeneous medium:

$$C = \text{const } \to \frac{\partial}{\partial x}\frac{1}{C} = \frac{\partial}{\partial y}\frac{1}{C} = \frac{\partial}{\partial z}\frac{1}{C} \equiv 0$$

and therefore we get the general solution to Euler's equations

$$\frac{1}{C}\frac{dx}{ds} = a_1 = \text{const}, \quad x = a_1Cs + x(0)\,;$$
$$\frac{1}{C}\frac{dy}{ds} = a_2 = \text{const}, \quad y = a_2Cs + y(0)\,;$$
$$\frac{1}{C}\frac{dz}{ds} = a_3 = \text{const}, \quad x = a_3Cs + z(0)\,.$$

Thus, rays in a homogeneous medium are straight lines. According to the definition of the arc length s we get

$$\begin{aligned} s &= \sqrt{(x-x(0))^2+(y-y(0))^2+(z-z(0))^2} = \\ &= Cs\sqrt{a_1^2+a_2^2+a_3^2} \longrightarrow \sqrt{a_1^2+a_2^2+a_3^2} = \frac{1}{C} \,. \end{aligned}$$

2.3 Hamiltonian form of the functional and Euler's equations

Consider for the sake of simplicity an 1D case, i.e.

$$T = \int_A^B L(\dot{x}, x) d\sigma \,.$$

We introduce a generalized pulse or generalized slowness through the formula $p = \partial L / \partial \dot{x}$ and from here we must find $\dot{x}$ as a function of p, i.e. $\dot{x} = \dot{x}(p)$.

The Hamiltonian function is defined by the formula

$$H = (p\dot{x} - L)\big|_{\dot{x}=\dot{x}(p)} \Rightarrow H = H(p, x) \,.$$

Inversely, $L = p\dot{x} - H$. In mechanics H means usually the energy of the mechanical system under consideration.

The expression for the functional now takes the form

$$T = \int_A^B L(\dot{x}, x) d\sigma = \int_A^B (p\dot{x} - H(p,x))\, d\sigma = \int_A^B p dx - H(p,x) d\sigma \,.$$

First variation of T. We introduce both variations of x and p

$$x \to x + \delta x \,, \quad p \to p + \delta p$$

and after calculations, very similar to the ones presented in the previous section, we obtain

$$\begin{aligned} \delta T &= \int_A^B \left(\delta p \dot{x} + p \delta \dot{x} - \frac{\partial H}{\partial p} \delta p - \frac{\partial H}{\partial x} \delta x \right) d\sigma = \\ &= p\delta x\big|_A^B + \int_A^B \left\{ \left(\dot{x} - \frac{\partial H}{\partial p} \right) \delta p - \left(\dot{p} + \frac{\partial H}{\partial x} \right) \delta x \right\} d\sigma \,. \end{aligned}$$

Euler's equations in Hamiltonian form follow from the latter expression for δT

$$\dot{x} = \frac{\partial H}{\partial p} \,, \quad \dot{p} = -\frac{\partial H}{\partial x} \,.$$

Thus, in Hamiltonian form we have a system of two ordinary differential equations of first order with respect to $x = x(\sigma)$ and $p = p(\sigma)$. In Lagrangian form, we had only one second order ordinary differential equation with respect to $x = x(\sigma)$:

$$\frac{d}{d\sigma} \frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} = 0 \,.$$

The extension to a 3D case is straightforward. For the Lagrangian $L = L(\dot{x}, \dot{y}, \dot{z}; x, y, z)$ we introduce three slownesses p_1, p_2, p_3 corresponding to the coordinates x, y, z, respectively,

$$p_1 = \frac{\partial L}{\partial \dot{x}} , \quad p_2 = \frac{\partial L}{\partial \dot{y}} , \quad p_3 = \frac{\partial L}{\partial \dot{z}} .$$

This usually nonlinear system of equations has to be solved with respect to $\dot{x}$, $\dot{y}$, $\dot{z}$. By inserting the solution into the formula

$$H = p_1 \dot{x} + p_2 \dot{y} + p_3 \dot{z} - L$$

we obtain the Hamiltonian function $H = H(p_1, p_2, p_3; x, y, z)$. The functional T takes the form of a curvilinear integral

$$T = \int_A^B L(\dot{x}, \dot{y}, \dot{z}; x, y, z) d\sigma = \int_A^B p_1 dx + p_2 dy + p_3 dz - H d\sigma$$

and its first variation reads

$$\begin{aligned}\delta T = \big(p_1 \delta x + p_2 \delta y + p_3 \delta z\big)\Big|_A^B + \int_A^B \Bigg\{ \left(\dot{x} - \frac{\partial H}{\partial p_1} \right) \delta p_1 + \\ + \left(\dot{y} - \frac{\partial H}{\partial p_2} \right) \delta p_2 + \left(\dot{z} - \frac{\partial H}{\partial p_3} \right) \delta p_3 - \left(\dot{p}_1 + \frac{\partial H}{\partial x} \right) \delta x - \\ - \left(\dot{p}_2 + \frac{\partial H}{\partial y} \right) \delta y - \left(\dot{p}_3 + \frac{\partial H}{\partial z} \right) \delta z \Bigg\} d\sigma .\end{aligned}$$

Euler's equations in Hamiltonian form derive from the above formula for δT

$$\begin{aligned}\dot{x} &= \frac{\partial H}{\partial p_1} , \quad \dot{p}_1 = -\frac{\partial H}{\partial x} , \\ \dot{y} &= \frac{\partial H}{\partial p_2} , \quad \dot{p}_2 = -\frac{\partial H}{\partial y} , \\ \dot{z} &= \frac{\partial H}{\partial p_3} , \quad \dot{p}_3 = -\frac{\partial H}{\partial z} .\end{aligned}$$

2.4 Solution of the eikonal equation in the case of a point source

We can describe a ray as a vector function

$$\vec{r}(\sigma) = x(\sigma)\vec{i} + y(\sigma)\vec{j} + z(\sigma)\vec{k'}$$

where $x(\sigma)$, $y(\sigma)$ and $z(\sigma)$ satisfy Euler's equations.

Consider a family of rays emanating from a point A, where a point source is located.

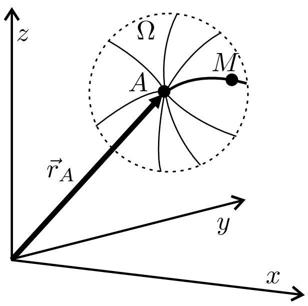

Figure 2.3: The central ray field regular in a domain Ω.

To this end we have to solve Euler's equations with the following initial data

$$\vec{r}(\sigma)\big|_{\sigma=0} = \vec{r}_A \; ; \quad \frac{d\vec{r}}{d\sigma}\Big|_{\sigma=0} = \vec{t}\,,$$

where $\vec{t}$ is a tangent vector to the ray at the point A, (if $\sigma = s$ then $|\vec{t}| = 1$, if σ is arbitrary then $|\vec{t}| \neq 1$). Suppose we solved the initial value problem for Euler's equations for an arbitrary direction of vector $\vec{t}$. We then obtain a family of rays which covers some region nearby a point A.

Definition: We say that this family of rays forms a regular field of rays in a domain Ω if for each point $M \in \Omega$ there is one and only one ray which starts at A and reaches M.

Suppose the family of rays is regular in Ω. In order to construct a solution of the eikonal equation in this case (in domain Ω) we have to perform the following procedure:

i) for each point $M \in \Omega$ we must find the ray which reaches M,

ii) then we must compute Fermat's integral $T = \int_A^M ds/C$ along this ray between points A and M. We thus obtain a function of position M which we denote by $\tau(M) = \tau(x, y, z)$, (x, y, z are coordinates of M). Evidently,

$$\tau(M) = \int_A^M \frac{ds}{C} = \int_A^M \frac{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}{C\big(x(\sigma), y(\sigma), z(\sigma)\big)} d\sigma\,.$$

This function $\tau(x, y, z)$ will satisfy the eikonal equation.

Proof. To this end we must calculate $\partial\tau/\partial x, \partial\tau/\partial y, \partial\tau/\partial z$ and substitute them in the eikonal equation $(\nabla\tau)^2 = 1/C^2$.

Consider the variation of T when A is fixed but M is mobile:

$$\begin{aligned} \delta T \;=\; & \left(\frac{\partial L}{\partial \dot{x}}\delta x + \frac{\partial L}{\partial \dot{y}}\delta y + \frac{\partial L}{\partial \dot{z}}\delta z\right)\bigg|_M + \int_A^M \left\{ \left(\frac{\partial L}{\partial x} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{x}}\right)\delta x + \right. \\ & \left. + \left(\frac{\partial L}{\partial y} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{y}}\right)\delta y + \left(\frac{\partial L}{\partial z} - \frac{d}{d\sigma}\frac{\partial L}{\partial \dot{z}}\right)\delta z \right\} d\sigma\,. \end{aligned} \tag{2.3}$$

As each time we compute T on a ray, Euler's equations are satisfied and because of that integral in (2.3) vanishes. On the other hand $d\tau = \delta T$ and therefore

$$d\tau \equiv \frac{\partial\tau}{\partial x}dx + \frac{\partial\tau}{\partial y}dy + \frac{\partial\tau}{\partial z}dz = \left(\frac{\partial L}{\partial \dot{x}}\delta x + \frac{\partial L}{\partial \dot{y}}\delta y + \frac{\partial L}{\partial \dot{z}}\delta z\right)\Big|_M .$$

Further,

$$dx = \delta x\big|_M,\ dy = \delta y\big|_M,\ dz = \delta z\big|_M ,$$

therefore

$$\left.\begin{aligned} \frac{\partial\tau}{\partial x} &= \frac{\partial L}{\partial \dot{x}} = \frac{\dot{x}}{C\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} \\ \frac{\partial\tau}{\partial y} &= \frac{\partial L}{\partial \dot{y}} = \frac{\dot{y}}{C\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} \\ \frac{\partial\tau}{\partial z} &= \frac{\partial L}{\partial \dot{z}} = \frac{\dot{z}}{C\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}} \end{aligned}\right\} \Rightarrow (\nabla\tau)^2 = \frac{\dot{x}^2+\dot{y}^2+\dot{z}^2}{C^2(\dot{x}^2+\dot{y}^2+\dot{z}^2)} = \frac{1}{C^2} . \quad (2.4)$$

Definition. Surfaces in 3D, defined by equation $\tau = \text{const}$, are called wave-fronts.

Wave fronts and rays are mutually orthogonal. Indeed, for $\tau = \text{const}$, evidently, $d\tau = 0$, i.e.

$$0 = \frac{\partial\tau}{\partial x}dx + \frac{\partial\tau}{\partial y}dy + \frac{\partial\tau}{\partial z}dz = (\nabla\tau, d\vec{r})$$

where $d\vec{r}$ belongs to a tangent plane to the surface $\tau = \text{const}$. It follows from here that gradτ is orthogonal to $\tau = \text{const}$, but taking into account (2.4) we get, for instance,

$$\frac{\partial\tau}{\partial x} = \frac{1}{C}\frac{dx}{\sqrt{\dot{x}^2+\dot{y}^2+\dot{z}^2}d\sigma} = \frac{1}{C}\frac{dx}{ds}$$

and similarly for $\partial\tau/\partial y$ and $\partial\tau/\partial z$.

Therefore

$$\text{grad}\tau \equiv \frac{\partial\tau}{\partial x}\vec{i} + \frac{\partial\tau}{\partial y}\vec{j} + \frac{\partial\tau}{\partial z}\vec{k'} = \frac{1}{C}\left(\frac{dx}{ds}\vec{i} + \frac{dy}{ds}\vec{j} + \frac{dz}{ds}\vec{k'}\right) = \frac{1}{C}\vec{t},$$

where $\vec{t}$ is a vector of unit length $|\vec{t}| = 1$ tangent to the ray.

2.5 Solution of the eikonal equation when an initial wave front is given

In this case, the ray field is formed by rays starting from each point of the surface $\tau = \tau_o = \text{const}$ in an orthogonal direction.

If this ray field is regular in some domain, then the solution of the eikonal equation $\tau(M) = \tau(x, y, z)$ in this domain reads

$$\tau(M) = \int_{M_o}^{M} \frac{ds}{C} + \tau_0 ,$$

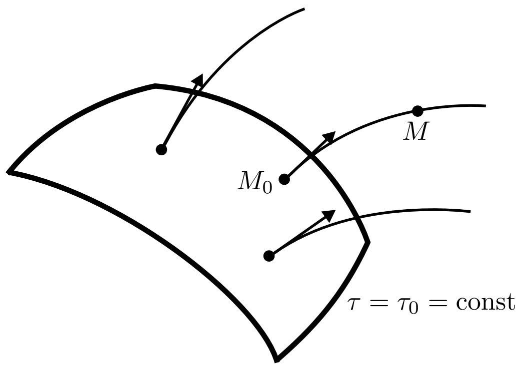

Figure 2.4: The ray field caused by a given initial wave front $\tau = \tau_0$.

where the integral is a curvilinear integral along that unique ray which starts at M_o and reaches M.

In a homogeneous medium this procedure coincides with Huygen's principle.

Each point on a wavefront $t = t_0$ is considered as a secondary source, which irradiates a spherical wave front that propagates during a time-interval Δt. An envelope to these secondary spherical waves describes the position of the wavefront at the moment $t = t_o + \Delta t$.

3

Solution of the transport equation

3.1 Ray coordinates

Suppose that the parameter σ along a ray is the eikonal τ, thus

$$d\tau = \frac{ds}{C} = \frac{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}{C} d\sigma$$

and respectively

$$\frac{d}{ds} = \frac{d}{Cd\tau} = \frac{1}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} \frac{d}{d\sigma} .$$

Taking this into account, Euler's equations for rays can be rewritten in terms of the new argument τ.

A family of rays in this case depends upon two parameters α, β, say, the angles of the spherical system of coordinates. So we can present them in the form

$$\vec{r} = \vec{r}(\tau, \alpha, \beta) .$$

For fixed α and β we have the ray given by the vector function

$$\vec{r} = x(\tau, \alpha, \beta)\vec{i} + y(\tau, \alpha, \beta)\vec{j} + z(\tau, \alpha, \beta)\vec{k'}$$

where τ indicates the position of a point on this ray. Now each point M in Ω can be described by its Cartesian coordinates x, y, z and by appropriate values of τ, α, β.

Definition. We say that τ, α, β form the ray coordinates in a domain Ω, where the central ray field is regular.

We can follow the same rationale in developing the ray coordinates if an initial position of the wave front is given. This time the parameters α and β specify the position of a point on the initial wave front $\tau = \tau_o$, from which the ray $\vec{r}(\tau, \alpha, \beta)$

starts in an orthogonal direction to this wave front. As previously, τ specifies the position of a point on the ray.

In a domain Ω where this ray field is regular τ, α, β form the ray coordinates.

Ray coordinates are formed in 2D by one parameter, say, α and eikonal τ: $\vec{r} = \vec{r}(\tau, \alpha)$.

Obviously, instead of τ we can take the arc length s along the ray or even an arbitrary parameter σ.

3.2 Auxiliary formulas

Functional determinants.

Consider two coordinate systems x, y, z and ξ, υ, ζ:

$$\begin{cases} x = x(\xi, \upsilon, \zeta) \\ y = y(\xi, \upsilon, \zeta) \\ z = z(\xi, \upsilon, \zeta) \end{cases} \tag{3.1}$$

Let us set the following question: is it possible to solve (3.1) in order to find inverse functions

$$\begin{cases} \xi = \xi(x, y, z) \\ \upsilon = \upsilon(x, y, z) \\ \zeta = \zeta(x, y, z) \end{cases} \quad ? \tag{3.2}$$

Assume that $x = y = z = 0$ corresponds to $\xi = \upsilon = \zeta = 0$, then in a vicinity of the origin we can present (3.1) approximately as follows

$$\begin{aligned} x &\simeq \frac{\partial x}{\partial \xi}\xi + \frac{\partial x}{\partial \upsilon}\upsilon + \frac{\partial x}{\partial \zeta}\zeta \\ y &\simeq \frac{\partial y}{\partial \xi}\xi + \frac{\partial y}{\partial \upsilon}\upsilon + \frac{\partial y}{\partial \zeta}\zeta \\ z &\simeq \frac{\partial z}{\partial \xi}\xi + \frac{\partial z}{\partial \upsilon}\upsilon + \frac{\partial z}{\partial \zeta}\zeta \,. \end{aligned} \tag{3.3}$$

Definition. The determinant of the system (3.3)

$$\Delta = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \upsilon} & \frac{\partial x}{\partial \zeta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \upsilon} & \frac{\partial y}{\partial \zeta} \\ \frac{\partial z}{\partial \xi} & \frac{\partial z}{\partial \upsilon} & \frac{\partial z}{\partial \zeta} \end{vmatrix}$$

is called functional determinant and is denoted as follows

$$\Delta = \frac{D(x, y, z)}{D(\xi, \upsilon, \zeta)} \,.$$

We know from linear algebra, that if $\Delta = D(x, y, z)/D(\xi, \upsilon, \zeta) \neq 0$ then the system of linear equations (3.3) can be solved solely with respect to ξ, υ, ζ.

Thus, it turns out that the functional determinants regulate the possibility to solve system (3.1) with respect to ξ, υ, ζ, i.e., to find inverse functions (3.2). Note that the complete answer to the question is known in mathematics as the theorem on a function given in implicit form.

Geometrical sense of the functional determinants.

Let us denote by $(\vec{a}, [\vec{b}, \vec{c}])$ the mixed vector product. We know from vector algebra, that the absolute value of the mixed product is the volume of the parallelepiped formed by the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Let us present a system of scalar equations (3.1) in vector form $\vec{r} = \vec{r}(\xi, \upsilon, \zeta)$. Obviously, if we fix, for instance, υ, ζ and vary only ξ we shall get a curve in 3D space: $\vec{r} = \vec{r}(\xi, \upsilon_0, \zeta_0)$, ($\upsilon_0, \zeta_0$ are fixed), and the differential of this vector function $d\vec{r} = (\partial\vec{r}/\partial\xi)d\xi$ is a vector tangent to this curve.

Thus we can construct 3 vectors

$$d_\xi\vec{r} = \frac{\partial\vec{r}}{\partial\xi}d\xi = \left(\frac{\partial x}{\partial\xi}\vec{i} + \frac{\partial y}{\partial\xi}\vec{j} + \frac{\partial z}{\partial\xi}\vec{k'}\right)d\xi$$

$$d_\upsilon\vec{r} = \frac{\partial\vec{r}}{\partial\upsilon}d\upsilon = \left(\frac{\partial x}{\partial\upsilon}\vec{i} + \frac{\partial y}{\partial\upsilon}\vec{j} + \frac{\partial z}{\partial\upsilon}\vec{k'}\right)d\upsilon$$

$$d_\zeta\vec{r} = \frac{\partial\vec{r}}{\partial\zeta}d\zeta = \left(\frac{\partial x}{\partial\zeta}\vec{i} + \frac{\partial y}{\partial\zeta}\vec{j} + \frac{\partial z}{\partial\zeta}\vec{k'}\right)d\zeta$$

Now for the element of volume dV in coordinates ξ, υ, ζ we obtain the following expression

$$dV = |(d_\xi\vec{r}, [d_\upsilon\vec{r}, d_\zeta\vec{r}])| = \left|\begin{vmatrix} \frac{\partial x}{\partial\xi} & \frac{\partial y}{\partial\xi} & \frac{\partial z}{\partial\xi} \\ \frac{\partial x}{\partial\upsilon} & \frac{\partial y}{\partial\upsilon} & \frac{\partial z}{\partial\upsilon} \\ \frac{\partial x}{\partial\zeta} & \frac{\partial y}{\partial\zeta} & \frac{\partial z}{\partial\zeta} \end{vmatrix} d\xi d\upsilon d\zeta\right| =$$

$$= \left\|\begin{matrix} \frac{\partial x}{\partial\xi} & \frac{\partial x}{\partial\upsilon} & \frac{\partial x}{\partial\zeta} \\ \frac{\partial y}{\partial\xi} & \frac{\partial y}{\partial\upsilon} & \frac{\partial y}{\partial\zeta} \\ \frac{\partial z}{\partial\xi} & \frac{\partial z}{\partial\upsilon} & \frac{\partial z}{\partial\zeta} \end{matrix}\right\| d\xi d\upsilon d\zeta = \left|\frac{D(x, y, z)}{D(\xi, \upsilon, \zeta)}\right| d\xi d\upsilon d\zeta\ .$$

Thus, the functional determinant $D(x, y, z)/D(\xi, \upsilon, \zeta)$ appears as a scalar factor for the element of volume dV in the new coordinate system ξ, υ, ζ. This can be regarded as the geometrical sense of functional determinants. Note that in the analysis, such a scalar factor is usually called Jacobian.

Definition of divergence.

Suppose we have a vector field

$$\vec{A}(x, y, z) = A_1\vec{i} + A_2\vec{j} + A_3\vec{k'}, \quad A_n = A_n(x, y, z), \quad n = 1, 2, 3.$$

Then in the Cartesian coordinates and only in these coordinates we have

$$\mathrm{div}\vec{A} = \frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z} \quad .$$

The definition of divergence, which does not depend upon a system of coordinates is the following. Consider a small body bounded by a closed surface S. Denote by ΔV the volume of the body and let $\vec{v}$ be the outgoing unit normal vector to the surface S. Then, by $(\vec{A}, \vec{v})|_s$ we denote the scalar product of vectors $\vec{A}$ and $\vec{v}$ calculated at points on S. Now the definition of div$\vec{A}$ follows

$$\mathrm{div}\vec{A} = \lim_{\Delta V \to 0} \frac{\int\int_{(S)} (\vec{A}, \vec{v})|_s ds}{\Delta V} \, .$$

The integral over the surface S in the numerator $\iint_{(S)} (\vec{A}, \vec{v})|_s ds$ is called the flux of vector field $\vec{A}$ through the closed surface S.

3.3 Geometrical spreading

Suppose a ray field is given in the form $\vec{r} = \vec{r}(\tau, \alpha, \beta)$, where τ is the eikonal, and α, β are ray parameters which specify the ray field.

Definition. We say that a set of rays for which the ray parameters α, β vary in the intervals

$$\alpha_0 \le \alpha \le \alpha_0 + d\alpha \, , \quad \beta_0 \le \beta \le \beta_0 + d\beta$$

forms a ray tube.

Consider a cross section of a ray tube by wave front $\tau = \tau_0$ and let us evaluate an area of this cross section $d\sum$

$$d\sum = \left|[d_\beta \vec{r}, d_\alpha \vec{r}]\right| = \left|\left[\frac{\partial \vec{r}}{\partial \beta}, \frac{\partial \vec{r}}{\partial \alpha}\right]\right| d\alpha d\beta \equiv J d\alpha d\beta \, ,$$

where

$$J = \left|\left[\frac{\partial \vec{r}}{\partial \beta}, \frac{\partial \vec{r}}{\partial \alpha}\right]\right|$$

is called the spreading of the ray tube, or geometrical spreading.

Geometrical sense of the spreading.

If the ray tube becomes wider J increases. If the ray tube becomes narrower and rays are focusing, for example, at a certain point J decreases and $J = 0$ precisely at that point.

Expression for J via the functional determinant.

Consider a piece of the ray tube cut by the wave front $\tau = \tau_0$ from one side and by $\tau = \tau_0 + d\tau$ from the other side.

For the volume of this part of the ray tube we have the following formula via the functional determinant

$$dV = \left|\frac{D(x,y,z)}{D(\tau,\alpha,\beta)}\right| d\tau d\alpha d\beta = \left|\frac{D(x,y,z)}{D(\tau,\alpha,\beta)}\right| \frac{1}{C} ds d\alpha d\beta$$

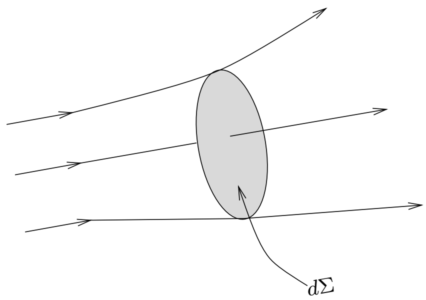

Figure 3.1: Cross section of a ray tube by wave front $\tau = \tau_0$.

because $d\tau = ds/C$ and s is the arc length along the rays.

On the other side, $dV = d\sum ds$ due to ds being the height of that piece of the ray tube (note, rays are orthogonal to wave fronts!). Here $d\sum$ is the area of the cross section and $d\sum = Jd\alpha d\beta$ in accordance with the definition of the spreading J. Therefore we finally get

$$dV = Jd\alpha d\beta ds\,.$$

By comparing both expressions with dV we obtain

$$J = \frac{1}{C}\left|\frac{D(x,y,z)}{D(\tau,\alpha,\beta)}\right|\,.$$

Example: An homogeneous medium.

Consider, for the sake of simplicity, a 2D homogeneous medium and a point source. In this case the rays are straight lines, the wave fronts τ = const are circles. Let the ray parameter α be the angle of the polar coordinate system. By ρ we denote the radius of the curvature of the wave front $\tau = \tau_o$.

Then, bearing geometry in mind, for an arbitrary wave front $\tau = \tau_1$, $\tau_1 > \tau_o$, we obtain

$$\frac{d\sum}{d\sum_0} \equiv \frac{Jd\alpha}{J_0 d\alpha} = \frac{[\rho + C(\tau_1 - \tau_0)]d\alpha}{\rho d\alpha} = \frac{\rho + C(\tau_1 - \tau_0)}{\rho}$$

due to $\tau = s/C + \text{const}$, where s is the arc length of the rays.

It follows from the above that the following formula for the spreading $J(\tau)$ as a function of τ holds true

$$J(\tau) = J(\tau_0)\frac{\rho + C(\tau - \tau_0)}{\rho}\,.$$

Similar calculations can be carried out in a 3D homogeneous medium. A final result in the case reads

$$J(\tau) = J(\tau_0)\frac{[\rho_1 + C(\tau - \tau_0)][\rho_2 + C(\tau - \tau_0)]}{\rho_1\rho_2}\,,$$

where ρ_1 and ρ_2 are the main radii of the curvature of the initial wave front $\tau = \tau_o$.

3.4 Solution of transport equations

Consider first the transport equation for the main amplitude A_0

$$2(\nabla\tau, \nabla A_0) + A_0 \Delta\tau = 0 \, .$$

This is a partial differential equation, but it can be presented as an ordinary differential equation along a ray. To this end, we have to recollect a definition of the derivative along a given curve. Let

$$x = x(s), y = y(s), z = z(s)$$

be a curve in 3D (or we can simply imagine a ray) and consider a function $f = f(x, y, z)$. Derivative along the curve is defined by the formula

$$\frac{df}{ds} = \frac{\partial f}{\partial x}\frac{dx}{ds} + \frac{\partial f}{\partial y}\frac{dy}{ds} + \frac{\partial f}{\partial z}\frac{dz}{ds} = (\nabla f, \vec{t}) \, ,$$

where $\vec{t}$ is a unit vector tangent to the curve.

Now we have to take into account that $\nabla\tau = \vec{t}/C$, where $\vec{t}$ is a unit tangent vector to a ray, therefore

$$(\nabla\tau, \nabla A_0) = \left(\nabla A_0, \frac{\vec{t}}{C}\right) = \frac{1}{C}(\nabla A_0, \vec{t}) = \frac{1}{C}\frac{dA_0}{ds} \, .$$

Instead of s, we can use the eikonal τ as a parameter along the ray. In this case we get

$$ds = C d\tau \to \frac{d}{ds} = \frac{1}{C}\frac{d}{d\tau}$$

and therefore

$$(\nabla\tau, \nabla A_0) = \frac{1}{C}\frac{dA_0}{ds} = \frac{1}{C^2}\frac{dA_0}{d\tau} \, .$$

Thus, finally we obtain

$$2(\nabla\tau, \nabla A_0) + A_0 \Delta\tau = \frac{2}{C^2}\frac{dA_0}{d\tau} + A_0 \Delta\tau = 0$$

which is a first order ordinary differential equation along the ray.

Subsequently, we have to calculate $\Delta\tau$ on the ray.

We know that $\Delta\tau = \text{div grad } \tau$. So this time the vector field is formed by grad $\tau = \vec{t}/C$. Then, we take a small body ΔV made of a piece of a ray tube and denote by $\vec{\nu}$ the external normal vector of unit length to the surface S of the body. Apparently, on the lateral surface of the ray tube $\vec{\nu} \perp \vec{t}$ and therefore

$$(\vec{A}, \vec{\nu}) = \frac{1}{C}(\vec{t}, \vec{\nu}) = 0 \, .$$

On the low bottom $\vec{\nu} \uparrow\downarrow \vec{t}$ and $(\vec{A}, \vec{\nu}) = -\frac{1}{C}$ while on the top section of the ray tube

$$(\vec{A}, \vec{\nu}) = \frac{1}{C}(\vec{t}, \vec{\nu}) = \frac{1}{C} \quad .$$

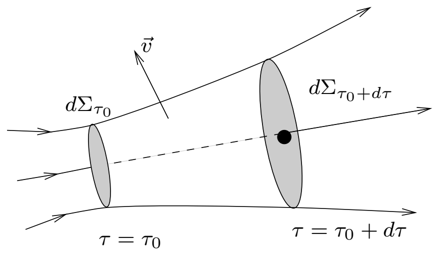

Figure 3.2: Calculation of $\Delta\tau$. A piece of a ray tube cut by two close wave fronts.

Hence,

$$\iint_{(S)} (\vec{A}, \vec{\nu})|_s \, dS \simeq -\frac{1}{C} d\sum_{\tau} + \frac{1}{C} d\sum_{\tau+d\tau} = \left(\frac{1}{C} J \bigg|_{\tau+d\tau} - \frac{1}{C} J \bigg|_{\tau} \right) d\alpha d\beta \, .$$

Then, for the element of volume dV we have

$$dV = \left| \frac{D(x,y,z)}{D(\tau,\alpha,\beta)} \right| d\tau d\alpha d\beta = CJ d\tau d\alpha d\beta$$

and therefore

$$\Delta\tau = \lim_{\Delta V \to 0} \frac{\left(\frac{1}{C} J|_{\tau+d\tau} - \frac{1}{C} J|_{\tau} \right)}{CJd\tau} = \frac{1}{CJ} \frac{d}{d\tau} \left(\frac{J}{C} \right) \quad .$$

Thus, finally we arrive at the following expression

$$2(\nabla\tau, \nabla A_0) + A_0 \Delta\tau = 0 \Rightarrow \frac{2}{C^2} \frac{dA_0}{d\tau} + \frac{A_0}{CJ} \frac{d}{d\tau} \left(\frac{J}{C} \right) = 0 \, .$$

The solution of this ordinary differential equation reads

$$\begin{aligned} \frac{dA_0}{A_0} &= -\frac{1}{2} \frac{1}{J/C} d(J/C) \to d \ln A_0 = \\ &= -\frac{1}{2} d \ln(J/C) \to \ln A_0 = -\frac{1}{2} \ln(J/C) + \ln \text{const} \to A_0 = \frac{\text{const}}{\sqrt{\frac{1}{C} J}} \, . \end{aligned}$$

We present the general solution of the transport equation for the main amplitude A_0 of the ray series in the following form

$$A_0 = \frac{\psi_0(\alpha,\beta)}{\sqrt{\frac{1}{C} J}} \, ,$$

where $\psi_o(\alpha, \beta)$ is a constant of integration with respect to τ, so it may only depend upon α, β !

The transport equations of a higher order read

$$2(\nabla\tau, \nabla A_{n+1}) + A_{n+1}\Delta\tau = \Delta A_n \quad , n = -1, 0, 1 \cdots ;\ A_{-1} \equiv 0,$$

and can be presented in the form

$$\frac{2}{C^2}\frac{dA_{n+1}}{d\tau} + \frac{A_{n+1}}{CJ}\frac{d}{d\tau}\left(\frac{J}{C}\right) = \Delta A_n \ ,$$

and

$$\frac{dA_{n+1}}{d\tau} + A_{n+1}\frac{1}{2}\frac{1}{J/C}\frac{d}{d\tau}(J/C) = \frac{C^2}{2}\Delta A_n. \tag{3.4}$$

The general solution reads

$$A_{n+1} = \sqrt{\frac{C}{J}}\left(\psi_{n+1}(\alpha, \beta) + \int_{\tau_0}^{\tau}\frac{C^2}{2}\sqrt{\frac{J}{C}}\Delta A_n d\tau\right) \ . \tag{3.5}$$

Explanations to formula(3.5). We have already got the general solution of the homogeneous equation (3.4). Indeed, it is

$$\tilde{A}_{n+1} = \frac{\text{const}}{\sqrt{\frac{1}{C}J}} \equiv \frac{\psi_{n+1}(\alpha, \beta)}{\sqrt{\frac{1}{C}J}} \ .$$

Now we have to find an arbitrary solution for the inhomogeneous equation (3.4) (i.e. with non zero right-hand side) in order to develop the general solution of this equation. To this end let us seek it in the form

$$\tilde{\tilde{A}}_{n+1} = U(\tau)\tilde{A}_{n+1}(\tau) \ ,$$

where $U(\tau)$ is an unknown function.

By inserting the latter formula in (3.4) we obtain

$$\tilde{A}_{n+1}\frac{dU}{d\tau} + U\frac{d\tilde{A}_{n+1}}{d\tau} + U\tilde{A}_{n+1}\frac{1}{2}\frac{d}{d\tau}(J/C) = \frac{C^2}{2}\Delta A_n \ .$$

Due to $\tilde{A}_{n+1}$ satisfying the homogeneous equation (and therefore canceling the two last terms on the left-hand side of the latter equation) we obtain for U the following result:

$$\tilde{A}_{n+1}\frac{dU}{d\tau} = \frac{C^2}{2}\Delta A_n \to U = \int_{\tau_0}^{\tau}\tilde{A}_{n+1}^{-1}\frac{C^2}{2}\Delta A_n d\tau \ .$$

This solution satisfies the following initial condition $U(\tau_o) = 0$. But we have to get any solution, therefore we may take this one with const $\equiv \psi_{n+1}(\alpha, \beta) = 1$ in the expression for $\tilde{A}_{n+1}$. The general solution then will take the form of (3.5).

Remark. The main problem in computation of A_1 is caused by the term

$$\Delta A_0 = \Delta \frac{\psi_0(\alpha, \beta)}{\sqrt{\frac{1}{C} J}} \, ,$$

under the integral in equation (3.5) for n=0.

In the numerical computations we face the problem of finding the second derivatives of the geometrical spreading on the ray under consideration.

4

Energy relations in the ray theory (in the wave equation perspective). Caustic problems of the ray method

4.1 On propagation of energy along ray tubes

We consider the wave equation

$$\frac{1}{C^2}\frac{\partial^2 W}{\partial t^2} - \Delta W = 0$$

and, within zero-order approximation of the ray method, have its approximate solution in the form

$$W = e^{-i\omega(t-\tau)} A_0 \, .$$

The wave field density of energy ρ_E for the wave equation is defined by the formula

$$\rho_E = \frac{1}{2}\left(\frac{1}{C^2}|W_t|^2 + |\nabla W|^2\right) \, .$$

Consider a ray tube

$$\alpha_0 \le \alpha \le \alpha_0 + d\alpha \, , \quad \beta_0 \le \beta \le \beta_0 + d\beta$$

and its elementary volume cut by the wave fronts $\tau = \tau_0$ and $\tau = \tau_0 + d\tau$ at a moment $t = t_0$. Let us calculate its position at a moment $t > t_0$.

Obviously, the equation of a wave front in the time domain is $t - \tau = \text{const}$. So, if at the moment t_0 we have $t_0 - \tau_0 = \text{const}$, then at the moment t we get $t - \tau = t_0 - \tau_0$ and therefore $\tau = \tau_0 + t - t_0$.

Respectively, if $t_0 - (\tau_0 + d\tau) = \text{const}$ at $t = t_0$, then for this wave front we have $t - \tau = t_0 - (\tau_0 + d\tau)$ and consequently $\tau = \tau_0 + d\tau + t - t_0$ at the moment $t = t$. Denote these volumes by dV_{t_0} and dV_t, respectively.

By calculating the energy of the wave field in these volumes in the main approximation we have

$$\begin{aligned} |W_t|^2 &= |-i\omega e^{-i\omega(t-\tau)}A_0|^2 = \omega^2|A_0|^2\,, \\ |\nabla W|^2 &= |+i\omega e^{-i\omega(t-\tau)}A_0\nabla\tau + e^{-i\omega(t-\tau)}\nabla A_0|^2 \simeq \omega^2(\nabla\tau)^2|A_0|^2 \\ &= \frac{\omega^2}{C^2}|A_0|^2 \end{aligned}$$

and therefore for the density of the energy we obtain the following approximate formula

$$\rho_E \simeq \frac{\omega^2}{C^2}|A_0|^2\,.$$

Hence, by taking into account that $dE = \rho_E dV$, we obtain

$$dE|_{t_0} \cong \frac{\omega^2}{C^2_{(M_0)}}|A_{0(M_0)}|^2 dV_{t_0} = \frac{\omega^2}{C^2_{(M_0)}}|A_{0(M_0)}|^2 C_{M_0}J_{M_0}d\tau d\alpha d\beta\,,$$

where all terms on the right-hand side are calculated at a point M_o in the elementary volume dV_{t_o}.

Respectively, we obtain at the moment t at the corresponding point M in dV_t

$$dE\big|_t = \frac{\omega^2}{C^2_{(M)}}|A_{0(M)}|^2 C_M J_M d\tau d\alpha d\beta\,.$$

By substituting the expression for the amplitude A_0, we obtain

$$dE\big|_{t_0} \simeq \frac{\omega^2}{C^2_{(M_0)}}\frac{|\psi_0(\alpha,\beta)|^2}{\dfrac{1}{C_{(M_0)}}J_{(M_0)}}C_{(M_0)}J_{(M_0)}d\tau d\alpha d\beta$$

and

$$dE|_t \simeq \frac{\omega^2}{C^2_{(M)}}\frac{|\psi_0(\alpha,\beta)|^2}{\dfrac{1}{C_{(M)}}J_{(M)}}C_{(M)}J_{(M)}d\tau d\alpha d\beta$$

and therefore

$$dE|_{t_0} = dE|_t = \omega^2|\psi_{0(\alpha,\beta)}|^2 d\tau d\alpha d\beta\,.$$

Note that the function $\psi_{0(\alpha,\beta)}$ is constant along the rays with the possibility of varying only from one ray to another.

The latter result demonstrates the following well known statement in the ray theory: the energy of the wave field in the zero order ray approximation propagates along ray tubes and it has no transversal diffusion across the ray tubes.

4.2 Energy relations for the wave equation and the vector of energy flow. Law of energy conservation

Consider first, for the sake of simplicity, the 1D wave equation

$$\frac{1}{C^2}\frac{\partial^2 W}{\partial t^2} - \frac{\partial^2 W}{\partial x^2} = 0$$

and assume that W is a real solution of this equation, then for the density of energy ρ_E we have by definition

$$\rho_E = \frac{1}{2}\frac{1}{C^2}\left(\frac{\partial W}{\partial t}\right)^2 + \frac{1}{2}\left(\frac{\partial W}{\partial x}\right)^2 .$$

By differentiating ρ_E with respect to time t we obtain

$$\frac{\partial \rho_E}{\partial t} = \frac{1}{C^2}\frac{\partial W}{\partial t}\frac{\partial^2 W}{\partial t^2} + \frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x \partial t} .$$

Further, after multiplying the wave equation by $\partial W/\partial t$ we can develop it as follows

$$\begin{aligned}\frac{1}{C^2}\frac{\partial W}{\partial t}\frac{\partial^2 W}{\partial t^2} - \frac{\partial W}{\partial t}\frac{\partial^2 W}{\partial x^2} &= \\ &= \frac{1}{C^2}\frac{\partial W}{\partial t}\frac{\partial^2 W}{\partial t^2} + \frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x \partial t} - \frac{\partial W}{\partial x}\frac{\partial^2 W}{\partial x \partial t} - \frac{\partial W}{\partial t}\frac{\partial^2 W}{\partial x^2} = \\ &= \frac{\partial \rho_E}{\partial t} - \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial t}\frac{\partial W}{\partial x}\right) = 0 .\end{aligned}$$

Apparently, for 3D we have

$$\rho_E = \frac{1}{2}\frac{1}{C^2}\left(\frac{\partial W}{\partial t}\right)^2 + \frac{1}{2}(\nabla W, \nabla W)$$

and get

$$\frac{\partial \rho_E}{\partial t} - \frac{\partial}{\partial x}\left(\frac{\partial W}{\partial t}\frac{\partial W}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial W}{\partial t}\frac{\partial W}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial W}{\partial t}\frac{\partial W}{\partial z}\right) = \frac{\partial \rho_E}{\partial t} + \operatorname{div} \vec{S} = 0 ,$$

where by vector $\vec{S}$ we denote

$$\vec{S} = -\frac{\partial W}{\partial t} \operatorname{grad} W .$$

Definition. $\vec{S}$ is called the vector of energy flow.

Thus, from the wave equation

$$\frac{1}{C^2}\frac{\partial^2 W}{\partial t^2} - \Delta W = 0$$

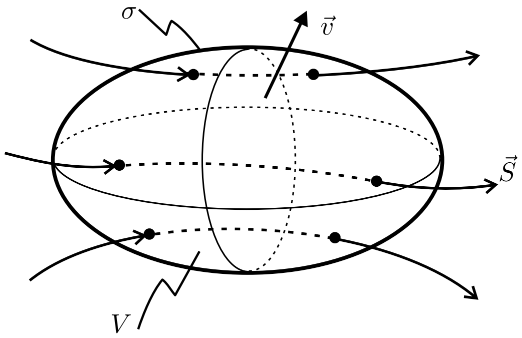

Figure 4.1: Vector of energy flow $\vec{S}$ and the law of energy conservation in a volume V bounded by a closed surface σ.

we deduce the following result

$$\frac{\partial \rho_E}{\partial t} + \text{div}\ \vec{S} = 0\,.$$

Let us take a body of volume V bounded by a closed surface σ, and integrate both sides of the latter equation over it:

$$\iiint\limits_V \frac{\partial \rho_E}{\partial t} dV + \iiint\limits_V \text{div}\ \vec{S}\, dV = \frac{\partial}{\partial t} \iiint\limits_V \rho_E dV + \iiint\limits_V \text{div}\ \vec{S}\, dV = 0\,.$$

Now we have to use the following auxiliary formula (Gauss-Ostrogradskii)

$$\iiint\limits_V \text{div}\ \vec{S}\, dV = \iint\limits_\sigma (\vec{S}, \vec{\nu})|_\sigma d\sigma \quad ,$$

where $\vec{\nu}$ is the unit vector of the outgoing normal to the surface σ and $d\sigma$ is the element of area on σ.

Then we finally obtain

$$\frac{\partial}{\partial t} \iiint\limits_V \rho_E dV = -\iint\limits_\sigma (\vec{S}, \vec{\nu})|_\sigma d\sigma\,.$$

The expression on the left-hand side of the latter equation describes the energy variation in the volume V for a unit interval of time, while on the right-hand side we have the total energy flow through a closed surface σ which bounds the volume. It follows from the equation that the increase (or decrease) of energy inside the volume is caused by the total flow of energy through the surface of the volume. This statement is known as the law of energy conservation.

In frames of the zero order approximation of the ray method we have for a real approximate solution W of the wave equation

$$W \simeq A_0 \cos\omega(t-\tau) \,,$$
$$\frac{\partial W}{\partial t} \simeq -\omega A_0 \sin\omega(t-\tau) \,,$$
$$\nabla W \simeq +\omega A_0 \nabla\tau \sin\omega(t-\tau)$$

and therefore

$$\vec{S} = -\frac{\partial W}{\partial t} \text{ grad } W \simeq \omega^2 A_0^2 \sin^2\omega(t-\tau)\text{grad}\tau = \frac{\omega^2 A_0^2}{C} \sin^2\omega(t-\tau)\,\vec{t}\,,$$

where $\vec{t}$ is the unit vector tangent to a ray. Thus, the direction of the energy flow vector coincides with the direction of the propagation of rays. It illuminates the important conception of rays of high-frequency approximation in the theory of wave propagation.

4.3 Caustic problems in the ray theory

The ray method solution in zero order approximation in the frequency domain reads

$$U = e^{i\omega\tau} A_0 = e^{i\omega\tau} \frac{\psi_{0(\alpha,\beta)}}{\sqrt{\frac{1}{C} J}} \,,$$

and for the geometrical spreading J we have the following formulas

$$J = \frac{1}{C} \left| \frac{D(x,y,z)}{D(\tau,\alpha,\beta)} \right| = \frac{1}{C} \left| (\vec{r}_\tau, [\vec{r}_\alpha \vec{r}_\beta]) \right|$$

which are convenient, if a family of rays is given in the form

$$\vec{r} = \vec{r}(\tau,\alpha,\beta) \quad .$$

Definition. A geometrical object in which at each point the geometrical spreading J vanishes is called caustic.

Thus, the amplitude A_0 gets singular on caustics, while the real wave field remains to be finite on the caustics. This means that the ray method does not properly describe the real wave field in a vicinity of caustics. This is precisely what we call caustic problems in the ray theory.

Point source. Suppose the corresponding family of rays is given in the form

$$\vec{r} = \vec{r}(\tau,\alpha,\beta) \,.$$

For each ray of the family we have

$$\vec{r}\big|_{\tau=0} = \vec{r}_A \,,$$

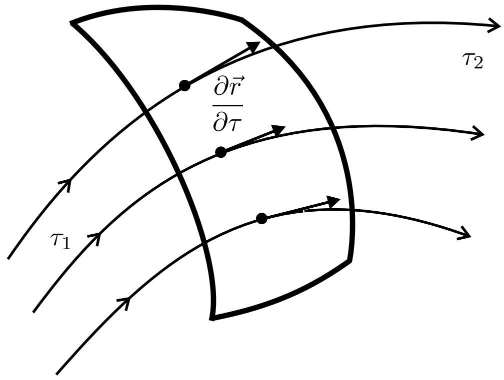

Figure 4.2: On a caustic surface eikonal τ becomes a function of the ray parameters α, β; τ_1 and τ_2 are the eikonals of incident and outgoing rays, respectively.

where $\vec{r}_A$ denotes the position of the source. By differentiating the latter formula with respect to the ray parameters α and β we get

$$\vec{r}_\alpha\big|_{\tau=0} = \vec{r}_\beta\big|_{\tau=0} = 0$$

and therefore $J = 0$ at the point source. Hence, in general the ray method is not valid in a vicinity of a point source! There are few exceptions of the ray formula coinciding with the exact solution.

Caustic surface. Assume now that the family of rays $\vec{r} = \vec{r}(\tau, \alpha, \beta)$ in 3D has an envelope surface. It implies that each ray of the family, specified by the ray parameters α and β, touches the envelope surface at a point which corresponds to a certain value of the eikonal τ. Obviously, this value of τ varies from ray to ray and therefore on the surface the eikonal τ is a function of the ray parameters, i.e. $\tau = \tau(\alpha, \beta)$. By inserting this function into the equation of rays we obtain the vectorial equation of the envelope surface in the following form:

$$\vec{r} = \vec{r}(\tau_{(\alpha,\beta)}, \alpha, \beta)\ .$$

It is clear now that the three vectors

$$\frac{\partial \vec{r}}{\partial \tau}\ ; \quad \frac{d\vec{r}}{d\alpha} = \frac{\partial \vec{r}}{\partial \tau}\frac{\partial \tau}{\partial \alpha} + \frac{\partial \vec{r}}{\partial \alpha}\ ; \quad \frac{d\vec{r}}{d\beta} = \frac{\partial \vec{r}}{\partial \tau}\frac{\partial \tau}{\partial \beta} + \frac{\partial \vec{r}}{\partial \beta}$$

are located on a tangent plane to the envelope surface and therefore their mixed product vanishes, i.e.

$$\left(\frac{\partial \vec{r}}{\partial \tau}, \left[\frac{d\vec{r}}{d\beta}, \frac{d\vec{r}}{d\alpha}\right]\right) = 0\ .$$

Note that the vector $\partial \vec{r}/\partial \tau$ is tangent to the ray and therefore it is tangent to the envelope, while the two others belong to a tangent plane to the envelope by definition.

The left-hand side of the latter equation can be developed as follows

$$\left(\vec{r}_\tau, \left[\vec{r}_\tau \frac{\partial\tau}{\partial\beta} + \vec{r}_\beta,\ \vec{r}_\tau \frac{\partial\tau}{\partial\alpha} + \vec{r}_\alpha\right]\right) =$$

$$= \left(\vec{r}_\tau, [\vec{r}_\tau, \vec{r}_\tau]\frac{\partial\tau}{\partial\beta}\frac{\partial\tau}{\partial\alpha} + [\vec{r}_\tau, \vec{r}_\alpha]\frac{\partial\tau}{\partial\beta} + [\vec{r}_\beta, \vec{r}_\tau]\frac{\partial\tau}{\partial\alpha} + [\vec{r}_\beta, \vec{r}_\alpha]\right) =$$

$$= \frac{\partial\tau}{\partial\beta}(\vec{r}_\tau, [\vec{r}_\tau \vec{r}_\alpha]) + \frac{\partial\tau}{\partial\alpha}(\vec{r}_\tau, [\vec{r}_\beta, \vec{r}_\tau]) + (\vec{r}_\tau[\vec{r}_\beta, \vec{r}_\alpha]) = (\vec{r}_\tau, [\vec{r}_\beta, \vec{r}_\alpha])$$

due to three other mixed products being identically equal to zero.

It follows immediately from here that $J = 0$ at any point of the envelope surface and therefore the ray method is not valid in its vicinity.

Quite often an envelope of a family of rays is called caustic. Our definition is then more extended, but there is no contradiction between them.

2D case. Now we have $x = x(\tau, \alpha)$; $y = y(\tau, \alpha)$ or in vector form $\vec{r} = \vec{r}(\tau, \alpha)$. Then,

$$\left|\frac{D(x,y)}{D(\tau,\alpha)}\right| = \left\| \begin{matrix} x_\tau, & y_\tau \\ x_\alpha, & y_\alpha \end{matrix} \right\| = \left|[\vec{r}_\tau, \vec{r}_\alpha]\right| .$$

Let α specify a point on an envelope line of a family of rays, and τ as usual specifies a point on a ray. It is clear that vectors $\vec{r}_\tau$ and $\vec{r}_\alpha$ are tangent to the envelope line and therefore at each point of the envelope we have

$$[\vec{r}_\tau, \vec{r}_\alpha] = 0 \Rightarrow J = 0 .$$

Several methods have been suggested and developed in order to overcome the caustic problems of the ray theory. Historically the first one is known now as the method of wave catastrophes used in the wave propagation and scattering theory rather than in Geophysics. In Geophysics it is called the modified ray method.

To illustrate the main ideas of the method let us consider a smooth branch of a caustic (simple caustic!). Denote by τ_1 the eikonal of the incident rays and by τ_2 the eikonal of the outgoing rays to the caustic, see Fig. 4.2. It can be proved that in a vicinity of the caustic, they can be presented in the form

$$\tau_1 = \xi - \frac{2}{3}\mu^{3/2} , \quad \tau_2 = \xi + \frac{2}{3}\mu^{3/2}$$

where $\xi = \xi(x, y, z)$ and $\mu = \mu(x, y, z)$ are regular functions (they have no singularities near the caustic!) Obviously, $\tau_1 = \tau_2$ exactly on the caustic and therefore $\mu = 0$ on it.

A wave field W in a vicinity of the caustic is constructed in the form of an asymptotic series with respect to the large frequency ω

$$\begin{aligned} W \quad = \quad & e^{i\omega\xi}\left\{\left[A_o(x,y,z) + \frac{A_1(x,y,z)}{-i\omega} + \ldots\right] v(-\omega^{3/2}\mu) + \right. \\ & \left. + \ i\left[B_o(x,y,z) + \frac{B_1(x,y,z)}{-i\omega} + \ldots\right]\frac{v'(-\omega^{3/2}\mu)}{w^{1/3}}\right\} \end{aligned} \quad (4.1)$$

where $v(t)$ is an Airy function $\big(v(t) = \sqrt{\pi}A_i(t)$ and $v'(t) = dv/dt\big)$. Regular functions A_k, B_k, $k = 0, 1, 2, \ldots$ have to be found by inserting the series into the reduced wave equation (compare with the ray method). The asymptotic expansion (4.1) was first suggested and developed by Kravtsov (1964) and by Ludwig (1966).

Remark. The Airy function $v(t)$ satisfies the following equation

$$v'' - tv = 0$$

and can be presented by the integral

$$v(t) = \frac{1}{\sqrt{\pi}} \int_O^\infty \cos\left(ty + \frac{y^3}{3}\right) dy\,.$$

The asymptotic behavior of $v(t)$ is the following.

For a large and positive t we have

$$v(t) \simeq \frac{1}{2} t^{-1/4} \exp\left(-\frac{2}{3} t^{3/2}\right) \quad \text{as } t \to +\infty\,, \tag{4.2}$$

while for a negative t, it oscillates

$$v(-t) \simeq t^{-1/4} \sin\left(\frac{2}{3} t^{3/2} + \frac{\pi}{4}\right) \quad \text{as } t \to +\infty\,. \tag{4.3}$$

It follows from here that for a positive large μ (it means the point of observation is far from the caustic but on the light-side of it!) v can be replaced by its asymptotic (4.3), resulting in two ray series for the incident and outgoing rays. On the contrary, for $\mu < 0$ and $|\mu| \to +\infty$ when the point of observation is located in the caustic shadow the wave field (4.1) exponentially decreases according to the asymptotics (4.2).

Thus, the asymptotic series (4.1) allows the development of an approximate solution of the reduced wave equation, which has no singularity on the caustic. But to this end we have to know what kind of special function has to be involved in the asymptotic series (the *Airy* function corresponds only to a simple caustic and, actually, it is the simplest function among the functions of wave catastrophes).

In the mid 1960s another method was proposed and developed by Maslov which is now known as Maslov's method - see Maslov (1965), Maslov and Fedoryuk (1976). In fact, this method provides a regular procedure for choosing a particular function of wave catastrophes, which corresponds to the particular geometrical structure of the caustic, and gives an expression for the wave field in terms of manifold integral. The method requires a good knowledge of Hamiltonian mechanics and does not seem to be easy for application. Presently, there are many papers in theoretical geophysics dealing with Maslov's method.

In the beginning of the 1980s, Popov proposed a new method for the computations of wave field in high-frequency approximation which is now known as the Gaussian Beam method - see Popov (1981,1982). This method provides a uniform computational algorithm which does not depend on a particular structure of the caustic. It does not require any special function to be used in the computations. This method will be discussed later in Chapters 8 and 9.

For more details concerning these methods and comparisons among them see the review paper by Babich and Popov (1989).

4.4 On the computational algorithm of the ray theory

We start with the scalar wave equation

$$\frac{1}{C^2}\frac{\partial^2 W}{\partial t^2} - \Delta W = 0$$

and construct approximate solutions in the form of ray series

$$W \sim e^{-i\omega(t-\tau)} \sum_{n=0}^{\infty} \frac{A_n}{(-i\omega)^n} \ ,$$

where the eikonal τ satisfies the eikonal equation $(\nabla\tau, \nabla\tau) = 1/C^2$ and for the main amplitude A_0 we get the transport equation

$$2(\nabla\tau, \nabla A_0) + A_0 \Delta\tau = 0 \quad .$$

What kind of computations should be carried out in order to apply it to geophysical problems?

Firstly, we have to construct rays, i.e. to solve Euler's equations for Fermat's functional.

This is a system of ordinary differential equations and apart from a few particular cases it can be solved only numerically. Then we have to find the eikonal (or travel time) by integrating along these rays

$$\tau = \tau_0 + \int \frac{ds}{C} \ .$$

Here we face the so-called two point problem:

How can that ray which connects a source and a point of observation be found?

In order to solve the problem we can use, for example, the trial and error method, which is a time consuming procedure.

The next step consists in the computation of the amplitude A_0, i.e., the geometrical spreading J due to

$$A_0 = \frac{\psi_0(\alpha, \beta)}{\sqrt{\frac{1}{C} J}} \ .$$

For J we have different analytical expressions:

$$J = \frac{1}{C}\left|\frac{D(x,y,z)}{D(\tau,\alpha,\beta)}\right| = \frac{1}{C}\left|(\vec{r}_\tau, [\vec{r}_\alpha, \vec{r}_\beta])\right| \quad .$$

If we take into account that

$$\vec{r}_\tau = \frac{d\vec{r}}{d\tau} = \frac{d\vec{r}}{ds}\frac{ds}{d\tau} = C\frac{d\vec{r}}{ds}$$

and that the vector $d\vec{r}/ds$ is orthogonal to $\vec{r}_\alpha$ and $\vec{r}_\beta$ and has unit length, we also get

$$J = \left|[\vec{r}_\alpha, \vec{r}_\beta]\right| \quad .$$

Obviously, to compute J we have to know the vectors $\vec{r}_\alpha$ and $\vec{r}_\beta$ as functions of τ or s.

One of the old but direct approaches to the problem consists in estimating the area of the cross section of a ray tube (in 2D it is the distance between two close rays). But normally, this problem requires special algorithms.

All these computations may fail, if the observation point is situated on a caustic or in a vicinity of a caustic. In that case the ray method itself requires improvements.

Note: there are a number of papers in theoretical geophysics in which the eikonal and the transport equations are treated as partial differential equations and finite difference methods are applied to them.

5
The paraxial ray theory

The main aim of this chapter is to study the eikonal and the amplitude of a ray series in the vicinity of a given ray, i.e., in a ray tube.

5.1 Ray centered coordinates

Assume, we have a given ray in the form

$$\vec{r} = \vec{r}_o(s) = x(s)\vec{i} + y(s)\vec{j} + z(s)\vec{k}$$

where s is the arc length of the ray.

Let us introduce two mutually orthogonal unit vectors $\vec{e}_1(s)$ and $\vec{e}_2(s)$ which at any point s belong to the plane orthogonal to this ray. We subject them to the following differential equations as functions of s

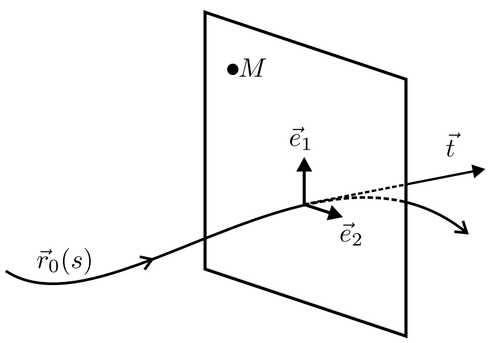

Figure 5.1: The ray centered coordinates in the vicinity of a ray $\vec{r}_o(s)$.

$$\frac{d\vec{e}_1}{ds} = \kappa_1(s)\vec{t}(s)\,, \quad \frac{d\vec{e}_2}{ds} = \kappa_2(s)\vec{t}(s)\,,$$

where $\vec{t} = d\vec{r}_o/ds$ is a unit vector tangent to the ray $\vec{r}_o(s)$ and the functions $\kappa_1(s)$ and $\kappa_2(s)$ are so far unknown. We shall fix them later.

Then, in the vicinity of this ray we can introduce local coordinates s, q_1, q_2 through the formula

$$\vec{r}_M = \vec{r}_0(s) + q_1\vec{e}_1(s) + q_2\vec{e}_2(s)\,,$$

where $\vec{r}_M$ is the radius-vector of a point M in a vicinity of a given ray $\vec{r}_o(s)$.

Let us calculate the element of length dS in this coordinate system. According to the definition

$$dS^2 = (d\vec{r}_M, d\vec{r}_M)$$

and

$$\begin{aligned} d\vec{r}_M &= \frac{d\vec{r}_0(s)}{ds}ds + dq_1\vec{e}_1(s) + dq_2\vec{e}_2(s) + q_1\frac{d\vec{e}_1}{ds}ds + q_2\frac{d\vec{e}_2}{ds}ds = \\ &= \vec{t}(1 + q_1\kappa_1 + q_2\kappa_2)ds + \vec{e}_1 dq_1 + \vec{e}_2 dq_2\,. \end{aligned}$$

Hence,

$$dS^2 = h^2 ds^2 + dq_1^2 + dq_2^2\,, \quad h = 1 + \kappa_1(s)q_1 + \kappa_2(s)q_2\,.$$

This coordinate system is regular and orthogonal in some vicinity of the central ray $\vec{r}_0(s)$. The latter fact follows from the expression for dS^2.

Now we can describe the rays which form a ray tube around this central ray $\vec{r}_0(s)$ by the following equations

$$q_1 = q_1(s) \quad \text{and} \quad q_2 = q_2(s)\,,$$

where functions $q_1(s)$, $q_2(s)$ have to satisfy Euler's equations.

5.2 Euler's equations in Hamiltonian form

We can now present Fermat's functional T in the following form

$$T = \int_{(A)}^{(B)} \frac{ds}{C} = \int_{(A)}^{(B)} \frac{\sqrt{h^2 + \dot{q}_1^2 + \dot{q}_2^2}}{C(s, q_1, q_2)} ds\,,$$

where $\dot{q}_i = dq_i/ds$, $i = 1, 2$, and velocity C is supposed to be a function of s, q_1, q_2.

In order to write Euler's equations in Hamiltonian form we need to introduce the slownesses p_1, p_2 with the formulas

$$p_i = \frac{1}{C}\frac{\partial}{\partial \dot{q}_i}\sqrt{h^2 + \dot{q}_1^2 + \dot{q}_2^2} = \frac{1}{C}\frac{\dot{q}_i}{\sqrt{h^2 + \dot{q}_1^2 + \dot{q}_2^2}}\,, \quad i = 1, 2. \tag{5.1}$$

Now we have to solve (5.1) with respect to $\dot{q}_i$, $i = 1, 2$. It follows from (5.1) that

$$\begin{aligned} \dot{q}_1^2(1 - C^2 p_1^2) - \dot{q}_2^2 C^2 p_1^2 &= C^2 p_1^2 h^2 \,, \\ -\dot{q}_1^2 C^2 p_2^2 + \dot{q}_2^2(1 - C^2 p_2^2) &= C^2 p_2^2 h^2 \,. \end{aligned}$$

Suppose that the determinant of this linear system is not equal to zero, i.e.,

$$\Delta = 1 - C^2(p_1^2 + p_2^2) \neq 0 \quad ,$$

then the unique solution reads

$$\begin{aligned} \dot{q}_1^2 &= \frac{C^2 p_1^2}{\Delta} h^2 \to \dot{q}_1 = \frac{C p_1 h}{\sqrt{\Delta}} \,, \\ \dot{q}_2^2 &= \frac{C^2 p_2^2}{\Delta} h^2 \to \dot{q}_2 = \frac{C p_2 h}{\sqrt{\Delta}} \,. \end{aligned}$$

For a Hamiltonian function H we get

$$\begin{aligned} H &= \left(\sum_{i=1}^{2} p_i \dot{q}_i - L \right) \Bigg|_{\dot{q}_i = \dot{q}_i(p_1, p_2)} = \\ &= \frac{Ch}{\sqrt{\Delta}} \big(p_1^2 + p_2^2\big) - \frac{1}{C} \sqrt{h^2 + \frac{C^2 h^2}{\Delta} \big(p_1^2 + p_2^2\big)} = \\ &= \frac{Ch}{\sqrt{\Delta}} \big(p_1^2 + p_2^2\big) - \frac{h}{C\sqrt{\Delta}} \sqrt{1 - C^2\big(p_1^2 + p_2^2\big) + C^2\big(p_1^2 + p_2^2\big)} = \\ &= -\frac{h}{C\sqrt{\Delta}} \big(1 - C^2(p_1^2 + p_2^2)\big) = -\frac{h}{C} \sqrt{1 - C^2\big(p_1^2 + p_2^2\big)} \,. \end{aligned}$$

Fermat's integral in Hamiltonian form now reads

$$T = \int_{(A)}^{(B)} \frac{ds}{C} = \int_{(A)}^{(B)} p_1 dq_1 + p_2 dq_2 - H(s, q_1, q_2, p_1, p_2) ds$$

and Hamiltonian equations have the form

$$\frac{d}{ds} q_i = \frac{\partial H}{\partial p_i} \,, \quad \frac{d}{ds} p_i = -\frac{\partial H}{\partial q_i} \,, \quad i = 1, 2 \,. \tag{5.2}$$

Note: the functions $\kappa_i(s)$, $i = 1, 2$, involved in the coordinate system s, q_1, q_2 are not fixed yet. In order to fix them we have to take into account that $\vec{r}_0(s)$ is a ray. That means, on one hand, that its equations in the ray centered coordinates are the following

$$q_1(s) = 0 \,, \quad q_2(s) = 0 \tag{5.3}$$

for an arbitrary s. By differentiating them on s we get $\dot{q}_1(s) = 0$, $\dot{q}_2(s) = 0$ and therefore $p_1(s) = 0$, $p_2(s) = 0$.

On the other hand, being a ray, $\vec{r}_0(s)$ has to satisfy Euler's equations and therefore the functions

$$q_1(s) = q_2(s) = 0 \quad \text{and} \quad p_1(s) = p_2(s) = 0$$

are solutions of Hamiltonian equations (5.2). By inserting them into equations (5.2) we conclude that the following formulas hold true along the central ray

$$\left.\frac{\partial H}{\partial q_j}\right|_{q_i=0,p_i=0} = 0, \quad \left.\frac{\partial H}{\partial p_j}\right|_{q_i=0,p_i=0} = 0, \quad j = 1, 2\,. \tag{5.4}$$

By inserting expression for H into (5.4) we obtain

$$\begin{aligned}\left.\frac{\partial H}{\partial q_j}\right|_{q_i=0,p_i=0} &= - \left.\left(\frac{1}{C}\frac{\partial h}{\partial q_j}\right)\right|_{q_i=0} + \left.\left(\frac{h}{C^2}\frac{\partial C}{\partial q_j}\right)\right|_{q_i=0} - \\ &- \left.\left(\frac{h}{C}\frac{-(p_1^2+p_2^2)C\dfrac{\partial C}{\partial q_j}}{\sqrt{1-C^2(p_1^2+p_2^2)}}\right)\right|_{q_i=0,p_i=0} = \\ &- \frac{1}{C(s,0,0)}\kappa_j(s) + \frac{1}{C^2(s,0,0)}\left.\frac{\partial C}{\partial q_j}\right|_{q_1=q_2=0} = 0, \quad j = 1, 2\end{aligned}$$

and therefore

$$\kappa_j(s) = \frac{1}{C(s,0,0)}\left.\frac{\partial C}{\partial q_j}\right|_{q_1=q_2=0}, \quad j = 1, 2. \tag{5.5}$$

Thus, functions κ_1 and κ_2 are fixed now by equations (5.5).

Remark1. Connections between Frenet's formulas and the ray centered coordinates.

Suppose we have an arbitrary smooth curve in 3D in the form $\vec{r} = \vec{r}(s)$ with s being the arc length of the curve. We can introduce three unit and mutually orthogonal vectors $\vec{t}$, $\vec{n}$, $\vec{b}$, where $\vec{t} = d\vec{r}/ds$ is a tangent vector to the curve, $\vec{n}$ is a vector of the main normal to the curve and $\vec{b} = [\vec{t}, \vec{n}]$ is a vector of the binormal to the curve.

For derivatives of these vectors with respect to s the following Frenet's formulas hold

$$\frac{d\vec{t}}{ds} = K\vec{n}\,, \quad \frac{d\vec{n}}{ds} = T\vec{b} - K\vec{t}, \quad \frac{d\vec{b}}{ds} = -T\vec{n}\,, \tag{5.6}$$

where $K = K(s)$ is the curvature of the curve and $T = T(s)$ is its torsion. Based on these vectors we can introduce other local coordinates, say, s, n, b by the formula

$$\vec{r}_M = \vec{r}_0(s) + n\,\vec{n}(s) + b\,\vec{b}(s)\,. \tag{5.7}$$

Assume now that the smooth curve is a ray. With the formula (5.7) we then obtain new coordinates in the vicinity of the ray, but unlike the ray centered coordinates s, q_1, q_2 this coordinate system will not be orthogonal! Indeed, we get from (5.7)

$$\begin{aligned} d\vec{r}_M &= \vec{t}ds + \vec{n}(s)dn + \vec{b}(s)db + n\frac{d\vec{n}}{ds}ds + b\frac{d\vec{b}}{ds}ds = \\ &= ds[\vec{t} + n(T\vec{b} - K\vec{t}) + b(-T\vec{n})] + \vec{n}(s)dn + \vec{b}(s)db \end{aligned}$$

and further

$$dS^2 = (d\vec{r}_M, d\vec{r}_M) = (1-nK)^2 ds^2 + (nTds + db)^2 + (dn - bTds)^2 = \\ \big[(1-nK)^2 + n^2T^2 + b^2T^2\big]ds^2 + db^2 + dn^2 + 2nTdsdb - 2bTdnds\,.$$

Because of the presence of the two last terms in this formula the coordinates s,n,b are not orthogonal in the vicinity of the central ray, simply per definition. But a non-orthogonal system is much less convenient for theoretical investigations.

We have introduced two other unit and mutually orthogonal vectors $\vec{e}_1(s)$, $\vec{e}_2(s)$ on the normal plane to the central ray $\vec{r}_0(s)$. Let us find the connection between them and $\vec{n}(s)$, $\vec{b}(s)$.

Introduce

$$\theta(s) = \int_{s_0}^{s} T(s)ds + \theta_0\,,$$

where $T(s)$ is the torsion of the ray, and then consider

$$\begin{aligned} \vec{e}_1(s) &= \vec{n}(s)\cos\theta(s) - \vec{b}(s)\sin\theta(s)\,, \\ \vec{e}_2(s) &= \vec{n}(s)\sin\theta(s) + \vec{b}(s)\cos\theta(s)\,. \end{aligned} \tag{5.8}$$

By differentiating the equations (5.8) with respect to s and taking into account Frenet's formulas (5.6) we obtain

$$\begin{aligned} \frac{d\vec{e}_1}{ds} &= -\vec{n}\sin\theta\,T + \cos\theta\frac{d\vec{n}}{ds} - \vec{b}\cos\theta\,T - \sin\theta\frac{d\vec{b}}{ds} = \\ &= -\vec{n}\sin\theta\,T + \cos\theta(T\vec{b} - K\vec{t}) - \vec{b}\cos\theta\,T + \sin\theta\,T\vec{n} = -K\vec{t}\cos\theta \end{aligned}$$

and, respectively,

$$\frac{d\vec{e}_2}{ds} = -K\sin\theta\,\vec{t}\,.$$

It follows from the latter formulas that derivatives of $\vec{e}_1$ and $\vec{e}_2$ have nonzero projections only on vector $\vec{t}$. This means precisely that during propagation along the central ray, both vectors $\vec{e}_1$, $\vec{e}_2$ do not rotate unlike vectors $\vec{n}$ and $\vec{b}$!

By comparing the latter formulas with equations (5.5) we arrive at the following result

$$\begin{aligned} \kappa_1(s) &= \frac{1}{C}\left.\frac{\partial C}{\partial q_1}\right|_{q_1=q_2=0} = -K(s)\cos\theta(s)\,, \\ \kappa_2(s) &= \frac{1}{C}\left.\frac{\partial C}{\partial q_2}\right|_{q_1=q_2=0} = -K(s)\sin\theta(s)\,, \end{aligned} \tag{5.9}$$

which hold true along the ray. For example, the curvature of the ray depends upon velocity and its derivatives as follows

$$K^2(s) = \left[\left(\frac{1}{C} \frac{\partial C}{\partial q_1} \right)^2 + \left(\frac{1}{C} \frac{\partial C}{\partial q_2} \right)^2 \right]_{q_1=q_2=0} .$$

5.3 Equations in variations

Consider now rays which form a ray tube centered on the ray $\vec{r}_0(s)$. They should be close to it and therefore q_1, q_2 along with p_1 and p_2 have to be small. Hence, we can simplify Hamiltonian equations (5.2) just by saving only the linear terms in decomposition its right-hand sides in power series on q and p. To this end let us expand H by using Taylor series.

We have, obviously,

$$\sqrt{1 - C^2(p_1^2 + p_2^2)} = 1 - \frac{1}{2} C_0^2 (p_1^2 + p_2^2) + \dots ,$$

and

$$\frac{1}{C} = \frac{1}{C_0} - \frac{1}{C_0^2} \frac{\partial C}{\partial q_1}\bigg|_{q_1=q_2=0} q_1 - \frac{1}{C_0^2} \frac{\partial C}{\partial q_2}\bigg|_{q_1=q_2=0} q_2 +$$
$$\frac{1}{2} \frac{\partial^2 C^{-1}}{\partial q_1^2}\bigg|_{q_1=q_2=0} q_1^2 + \frac{\partial^2 C^{-1}}{\partial q_1 \partial q_2}\bigg|_{q_1=q_2=0} q_1 q_2 + \frac{1}{2} \frac{\partial^2 C^{-1}}{\partial q_2^2}\bigg|_{q_1=q_2=0} q_2^2 + \dots ,$$

where by C_o we denote velocity C computed on the central ray $\vec{r}_o(s)$, i.e., $C_o \equiv C|_{q_1=q_2=0} = C_o(s)$.

Then,

$$H = -\frac{h}{C} \sqrt{1 - C^2(p_1^2 + p_2^2)} =$$
$$= -(1 + \kappa_1 q_1 + \kappa_2 q_2) \left(1 - \frac{1}{2} C_0^2 (p_1^2 + p_2^2) + \cdots \right) \left(\frac{1}{C_0} - \frac{1}{C_0} \kappa_1 q_1 - \frac{1}{C_0} \kappa_2 q_2 + \right.$$
$$\left. + \frac{1}{2} \frac{\partial^2 C^{-1}}{\partial q_1^2}\bigg|_{q_1=q_2=0} q_1^2 + \frac{\partial^2 C^{-1}}{\partial q_1 \partial_2}\bigg|_{q_1=q_2=0} q_1 q_2 + \frac{1}{2} \frac{\partial^2 C^{-1}}{\partial q_2^2}\bigg|_{q_1=q_2=0} q_2^2 + \cdots \right) =$$
$$= -\frac{1}{C_0} + \frac{1}{2} C_0 (p_1^2 + p_2^2) + \frac{1}{C_0} (\kappa_1 q_1 + \kappa_2 q_2)^2 -$$
$$- \frac{1}{2} \left(\frac{\partial^2 C^{-1}}{\partial q_1^2}\bigg|_{q_1=q_2=0} q_1^2 + 2 \frac{\partial^2 C^{-1}}{\partial q_1 \partial_2}\bigg|_{q_1=q_2=0} q_1 q_2 + \frac{\partial^2 C^{-1}}{\partial q_2^2}\bigg|_{q_1=q_2=0} q_2^2 \right) + \cdots$$

Now we have to develop second derivatives of C^{-1} with respect to q_1 and q_2:

$$\frac{\partial C^{-1}}{\partial q_1} = -1C^{-2}\frac{\partial C}{\partial q_1}\,, \quad \frac{\partial^2 C^{-1}}{\partial q_1^2} = 2C^{-3}\left(\frac{\partial C}{\partial q_1}\right)^2 - C^{-2}\frac{\partial^2 C}{\partial q_1^2} \Rightarrow$$
$$\left.\frac{\partial^2 C^{-1}}{\partial q_1^2}\right|_{q_1=q_2=0} = \frac{2}{C_0}\kappa_1^2 - \frac{1}{C_0^2}\left.\frac{\partial^2 C}{\partial q_1^2}\right|_{q_1=q_2=0}.$$

Accordingly,

$$\left.\frac{\partial^2 C^{-1}}{\partial q_2^2}\right|_{q_1=q_2=0} = \frac{2}{C_0}\kappa_2^2 - \frac{1}{C_0^2}\left.\frac{\partial^2 C}{\partial q_2^2}\right|_{q_1=q_2=0}$$

and

$$\begin{aligned}\left.\frac{\partial^2 C^{-1}}{\partial q_1 \partial q_2}\right|_{q_1=q_2=0} &= 2C_0^{-3}\frac{\partial C}{\partial q_1}\left.\frac{\partial C}{\partial q_2}\right|_{q_1=q_2=0} - \frac{1}{C_0^2}\left.\frac{\partial^2 C}{\partial q_1 \partial q_2}\right|_{q_1=q_2=0} = \\ &= \frac{2}{C_0}\kappa_1\kappa_2 - \frac{1}{C_0^2}\left.\frac{\partial^2 C}{\partial q_1 q_2}\right|_{q_1=q_2=0}.\end{aligned}$$

By taking it into account, we obtain

$$\begin{aligned}H = &-\frac{1}{C_0} + \frac{C_0}{2}(p_1^2+p_2^2) + \\ &+\frac{1}{2C_0}\left(\left.\frac{\partial^2 C}{\partial q_1^2}\right|_{q_1=q_2=0} q_1^2 + 2\left.\frac{\partial^2 C}{\partial q_1\partial q_2}\right|_{q_1=q_2=0} q_1q_2 + \left.\frac{\partial^2 C}{\partial q_2^2}\right|_{q_1=q_2=0} q_2^2\right) + \\ &+\cdots = H_0 + H_2 + \cdots\,,\end{aligned}$$

where $H_0 = -1/C_0$ and H_2 is the second order polynomial on q_1, q_2, p_1, p_2.

Thus, for the rays from a ray tube we obtain a linear system of ordinary differential equations

$$\frac{d}{ds}q_i = \frac{\partial H_2}{\partial p_i}\,, \quad \frac{d}{ds}p_i = -\frac{\partial H_2}{\partial q_i}\,, \quad i=1,2. \tag{5.10}$$

This system is called the equations in variations for the initial Hamiltonian system (5.2). By taking into account the expression for H_2 we ultimately get

$$\begin{aligned}&\frac{d}{ds}q_1 = C_0p_1\,, \quad \frac{d}{ds}p_1 = -\frac{1}{C_0^2}\left.\frac{\partial^2 C}{\partial q_1^2}\right|_{q_1=q_2=0} q_1 - \frac{1}{C_0^2}\left.\frac{\partial^2 C}{\partial q_1\partial q_2}\right|_{q_1=q_2=0} q_2\,, \\ &\frac{d}{ds}q_2 = C_0p_2\,, \quad \frac{d}{ds}p_2 = -\frac{1}{C_0^2}\left.\frac{\partial^2 C}{\partial q_1\partial q_2}\right|_{q_1=q_2=0} q_1 - \frac{1}{C_0^2}\left.\frac{\partial^2 C}{\partial q_2^2}\right|_{q_1=q_2=0} q_2\,.\end{aligned} \tag{5.11}$$

5.4 Properties of the solutions of the equations in variations

The properties of the solutions of the equations in variations listed below actually underlie the paraxial ray theory and the Gaussian Beam method as well. They are known among specialists in Hamiltonian mechanics but are not popular in geophysics yet. So one can find in geophysical literature, for instance, a description of the features of the Gaussian Beams without any mathematical basis, causing these features to be rather mysterious and not too reliable.

Let us write the equations in variations in matrix form. To this end we introduce the vector-column X

$$X = \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} .$$

The derivative of X is defined by

$$\dot{X} = \frac{d}{ds} X = \begin{pmatrix} \dot{q}_1 \\ \dot{q}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} .$$

Further, let us introduce the following matrix of second order

$$\mathbf{C} = \begin{pmatrix} \left.\dfrac{\partial^2 C}{\partial q_1 \partial q_1}\right|_{\substack{q_1=0\\ q_2=0}} & \left.\dfrac{\partial^2 C}{\partial q_1 \partial q_2}\right|_{\substack{q_1=0\\ q_2=0}} \\[2ex] \left.\dfrac{\partial^2 C}{\partial q_1 \partial q_2}\right|_{\substack{q_1=0\\ q_2=0}} & \left.\dfrac{\partial^2 C}{\partial q_2 \partial q_2}\right|_{\substack{q_1=0\\ q_2=0}} \end{pmatrix} \text{ or } \mathbf{C}_{ik} = \left.\frac{\partial^2 C}{\partial q_i \partial q_k}\right|_{q_1=q_2=0} , \; i,k = 1,2.$$

Then the equations in variations can be presented in the following form

$$\dot{X} = \begin{pmatrix} 0 & C_0 \mathbf{E} \\ -\dfrac{1}{C_0^2}\mathbf{C} & 0 \end{pmatrix} X ,$$

where

$$\mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ,$$

i.e. a unit matrix of second order. Indeed, in order to verify this formula it is sufficient to write it down in detail:

$$\begin{pmatrix} \dot q_1 \\ \dot q_2 \\ \dot p_1 \\ \dot p_2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & C_0 & 0 \\ 0 & 0 & 0 & C_0 \\ -\frac{1}{C_0^2}\mathbf{C}_{11} & \frac{-1}{C_0^2}\mathbf{C}_{12} & 0 & 0 \\ -\frac{1}{C_0^2}\mathbf{C}_{21} & \frac{-1}{C_0^2}\mathbf{C}_{22} & 0 & 0 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ p_1 \\ p_2 \end{pmatrix} . \tag{5.12}$$

Now we see that equations (5.11) coincide with equations (5.12) and therefore with equations (5.10).

In order to present equations (5.12) in a more convenient form let us introduce a matrix $\mathbf{J}$ of the fourth order

$$\mathbf{J} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{E} \\ -\mathbf{E} & 0 \end{pmatrix} , \quad \mathbf{E} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} .$$

It has the following properties:

$$\mathbf{J}^* = \begin{pmatrix} 0 & -\mathbf{E} \\ \mathbf{E} & 0 \end{pmatrix} = -\mathbf{J} , \quad \det \mathbf{J} = 1 .$$

$$\mathbf{J}^2 = \begin{pmatrix} 0 & \mathbf{E} \\ -\mathbf{E} & 0 \end{pmatrix} \begin{pmatrix} 0 & \mathbf{E} \\ -\mathbf{E} & 0 \end{pmatrix} = \begin{pmatrix} -\mathbf{E} & 0 \\ 0 & -\mathbf{E} \end{pmatrix} = -\mathbf{I} \left[\mathbf{I} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right] .$$

$$\mathbf{J}^*\mathbf{J} = -\mathbf{J}\mathbf{J} = -\mathbf{J}^2 = \mathbf{I} .$$

Denote by $\mathbf{H}$ the following symmetrical matrix

$$\mathbf{H} = \begin{pmatrix} \frac{1}{C_0^2}\mathbf{C} & 0 \\ 0 & C_0\mathbf{E} \end{pmatrix} ; \quad \mathbf{H}^* = \mathbf{H} .$$

Then the system of equations (5.11) can be written down in the desired form

$$\frac{d}{ds} X = \mathbf{J}\mathbf{H}X . \tag{5.13}$$

To check this result, it is enough to calculate the matrix product $\mathbf{JH}$

$$\mathbf{JH} = \begin{pmatrix} 0 & \mathbf{E} \\ -\mathbf{E} & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{C_0^2}\mathbf{C} & 0 \\ 0 & C_0\mathbf{E} \end{pmatrix} = \begin{pmatrix} 0 & C_0\mathbf{E} \\ -\frac{1}{C_0^2}\mathbf{C} & 0 \end{pmatrix}$$

and to compare it with formula (5.11) or (5.12).

In order to complete the list of additional notations, let us introduce the **J** - scalar product of two arbitrary solutions $X^{(1)}$ and $X^{(2)}$ of equations (5.13).

Define

$$(X^{(1)}, X^{(2)}) = \sum_{i=1}^{2} \left(q_i^{(1)} q_i^{(2)} + p_i^{(1)} p_i^{(2)} \right) ,$$

then for the **J**-scalar product $\left(\mathbf{J}X^{(1)}, X^{(2)}\right)$ we get

$$\left(\mathbf{J}X^{(1)}, X^{(2)}\right) = p_1^{(1)} q_1^{(2)} + p_2^{(1)} q_2^{(2)} - q_1^{(1)} p_1^{(2)} - q_2^{(1)} p_2^{(2)} .$$

Theorem. Let $X^{(1)}$ and $X^{(2)}$ be two arbitrary solutions of the equations in variations (5.13), then the **J** - scalar product of these solutions does not depend on s, i.e.,

$$\frac{d}{ds}(\mathbf{J}X^{(1)}, X^{(2)}) \equiv 0 .$$

Proof:

$$\begin{aligned}\frac{d}{ds}\left(\mathbf{J}X^{(1)}, X^{(2)}\right) &= \left(\mathbf{J}\frac{dX^{(1)}}{ds}, X^{(2)}\right) + \left(\mathbf{J}X^{(1)}, \frac{dX^{(2)}}{ds}\right) = \\ &= \left(\mathbf{J}^2\mathbf{H}X^{(1)}, X^{(2)}\right) + \left(\mathbf{J}X^{(1)}, \mathbf{J}\mathbf{H}X^{(2)}\right) = \\ &= -\left(\mathbf{H}X^{(1)}, X^{(2)}\right) + \left(\mathbf{J}^*\mathbf{J}X^{(1)}, \mathbf{H}X^{(2)}\right) = \\ &= -\left(\mathbf{H}X^{(1)}, X^{(2)}\right) + \left(X^{(1)}, \mathbf{H}X^{(2)}\right) = \\ &= -\left(\mathbf{H}X^{(1)}, X^{(2)}\right) + \left(\mathbf{H}^*X^{(1)}, X^{(2)}\right) \equiv 0\end{aligned}$$

due to $\mathbf{H}^* = \mathbf{H}$, see definition of **H**.

Consequence. Denote by $\mathbf{W}(s)$ the fundamental matrix of the system (5.13), i.e., its columns are formed by linearly independent solutions of the system and at an initial point $s = s_0$ equation $\mathbf{W}(s_0) = \mathbf{I}$ holds. Then $\mathbf{W}(s)$ satisfies the following condition

$$\mathbf{W}^*(s)\mathbf{J}\mathbf{W}(s) = \mathbf{J} .$$

Proof. Let us take two arbitrary vector-columns $Z^{(1)}$ and $Z^{(2)}$ formed by arbitrary real numbers, then $X^{(1)}(s) = \mathbf{W}(s)Z^{(1)}$ and $X^{(2)} = \mathbf{W}(s)Z^{(2)}$ are solutions of the system (5.13):

$$\frac{d}{ds}X^{(j)}(s) = \frac{d\mathbf{W}(s)}{ds}Z^{(j)} = \mathbf{J}\mathbf{H}\mathbf{W}(s)Z^{(j)} = \mathbf{J}\mathbf{H}X^{(j)} \quad , j = 1, 2 .$$

Hence,

$$\begin{aligned}(\mathbf{J}X^{(1)}, X^{(2)}) = \text{const} &= (\mathbf{J}\mathbf{W}Z^{(1)}, \mathbf{W}Z^{(2)}) = (\mathbf{W}^*\mathbf{J}\mathbf{W}Z^{(1)}, Z^{(2)}) = \\ &= (\mathbf{W}^*(s_0)\mathbf{J}\mathbf{W}(s_0)Z^{(1)}, Z^{(2)}) = (\mathbf{I}\mathbf{J}\mathbf{I}Z^{(1)}, Z^{(2)}) = (\mathbf{J}Z^{(1)}, Z^{(2)})\end{aligned}$$

and it holds for arbitrary $Z^{(1)}$ and $Z^{(2)}$. It follows from here that

$$\mathbf{W}^*(s)\mathbf{J}\mathbf{W}(s) = \mathbf{J} .$$

Remark 1. Reduction of the equations (5.11) to a 2D case.

To reduce this system to 2D we may simply assume that, for instance, $q_2 = p_2 = 0$ and make $q_1 = q$, $p_1 = p$.

Thus we obtain the following system of differential equations

$$\frac{d}{ds}q = C_0 p\,, \quad \frac{dp}{ds} = -\frac{1}{C_0^2}\left.\frac{\partial^2 C}{\partial q_1^2}\right|_{q_1=0} q\,,$$

which is valid in a 2D case.

Remark 2. Reduction to a 2.5D case.

Assume the vector $\vec{e}_2(s)$ is orthogonal to a plane on which a central ray is placed and the velocity C does not vary in this direction, i.e., $C = C(s, q_1)$. In this case we have $\partial^2 C/\partial q_1 \partial q_2 = 0$ and $\partial^2 C/\partial q_2^2 = 0$ and therefore equations (5.11) take the form

$$\begin{cases} \dfrac{d}{ds}q_1 = C_0 p_1 \\ \dfrac{dp_1}{ds} = -\dfrac{1}{C_0^2}\left.\dfrac{\partial^2 C}{\partial q_1^2}\right|_{q_1=0} q_1 \end{cases} \qquad \begin{cases} \dfrac{d}{ds}q_2 = C_0 p_2 \\ \dfrac{d}{ds}p_2 = 0 \end{cases} .$$

It means that we get two independent systems for q_1 and p_1 and for q_2 and p_2. The second one can be solved immediately

$$p_2(s) = p_2(s_0) = \text{const and } q_2(s) = p_2(s_0)\int_{s_0}^{s} C_0 ds + q_2(s_0)\,.$$

5.5 An algorithm for the computation of the geometrical spreading

We have the following formulas for the amplitude A and the geometrical spreading J

$$A_0 = \frac{\psi_0(\alpha,\beta)}{\sqrt{\dfrac{1}{C}J}}\,, \quad J = \frac{1}{C}\left|\frac{D(x,y,z)}{D(\tau,\alpha,\beta)}\right|\,.$$

Let us introduce the ray centered coordinates s, q_1, q_2. By using the chain rule for functional determinants, we obtain

$$\left|\frac{D(x,y,z)}{D(\tau,\alpha,\beta)}\right| = \left|\frac{D(x,y,z)}{D(s,q_1,q_2)}\frac{D(s,q_1,q_2)}{D(\tau,\alpha,\beta)}\right| = h\left|\frac{D(s,q_1,q_2)}{D(\tau,\alpha,\beta)}\right|\,. \tag{5.14}$$

The reason why h appears in (5.14) can be explained as follows. The modulus of the functional determinant is the Jacobian. But in orthogonal coordinates the Jacobian is equal to the square root of the product of Lame's coefficients in the expression for dS^2. The ray centered coordinates are orthogonal and there is only one coefficient h^2 which is not equal to one.

Suppose we know the solutions of equations (5.11) for the rays in the form

$$q_1 = q_1(s, \alpha, \beta)\,, \quad q_2 = q_2(s, \alpha, \beta)\,, \tag{5.15}$$

where s is the arc length of the central ray. Assume that for the central ray $\alpha = \alpha_0$, $\beta = \beta_0$. In order to calculate the functional determinant remained in equation (5.14), we have to find s as a function of τ, α, β. To this end, consider the eikonal τ:

$$\tau = \int_{s_0}^{s} \frac{\sqrt{h^2 + \dot{q}_1^2 + \dot{q}_2^2}}{C(s, q_1, q_2)} ds\,. \tag{5.16}$$

By inserting in equation (5.16) formulas (5.15) we get

$$\tau = \tau(s, \alpha, \beta)\,. \tag{5.17}$$

In a vicinity of the central ray ($\alpha = \alpha_0$, $\beta = \beta_0$) the latter equation can be solved (in principle!) with respect to s:

$$s = s(\tau, \alpha, \beta)\,.$$

By taking it into account, we rewrite equations (5.15) in the following form

$$q_1 = q_1(s(\tau, \alpha, \beta), \alpha, \beta)\,; \quad q_2 = q_2(s(\tau, \alpha, \beta), \alpha, \beta)\,. \tag{5.18}$$

It follows from (5.18) that $q_j, j = 1, 2$, are now compound functions of the ray parameters α and β. Therefore for the derivatives we get

$$\begin{aligned}
&\frac{\partial q_j}{\partial \tau} = \frac{\partial q_j}{\partial s}\frac{\partial s}{\partial \tau} = \dot{q}_j \frac{\partial s}{\partial \tau}\,;\\
&\frac{\partial q_j}{\partial \alpha} = \frac{\partial q_j}{\partial s}\frac{\partial s}{\partial \alpha} + \frac{\partial q_j}{\partial \alpha} = \dot{q}_j \frac{\partial s}{\partial \alpha} + \frac{\partial q_j}{\partial \alpha};\\
&\frac{\partial q_j}{\partial \beta} = \dot{q}_j \frac{\partial s}{\partial \beta} + \frac{\partial q_j}{\partial \beta}\,, \quad j = 1, 2.
\end{aligned} \tag{5.19}$$

Bearing in mind equations (5.19) we obtain for the functional determinant in (5.14) the following expression

$$\frac{D(s, q_1, q_2)}{D(\tau, \alpha, \beta)} = \begin{vmatrix} \dfrac{\partial s}{\partial \tau} & \dfrac{\partial s}{\partial \alpha} & \dfrac{\partial s}{\partial \beta} \\ \dot{q}_1 \dfrac{\partial s}{\partial \tau} & \dot{q}_1 \dfrac{\partial s}{\partial \alpha} + \dfrac{\partial q_1}{\partial \alpha} & \dot{q}_1 \dfrac{\partial s}{\partial \beta} + \dfrac{\partial q_1}{\partial \beta} \\ \dot{q}_2 \dfrac{\partial s}{\partial \tau} & \dot{q}_2 \dfrac{\partial s}{\partial \beta} + \dfrac{\partial q_2}{\partial \beta} & \dot{q}_2 \dfrac{\partial s}{\partial \beta} + \dfrac{\partial q_2}{\partial \beta} \end{vmatrix} =$$

which can be developed as follows

$$= \frac{\partial s}{\partial \tau}\frac{\partial s}{\partial \alpha}\frac{\partial s}{\partial \beta}\begin{vmatrix} 1 & 1 & 1 \\ \dot{q}_1 & \dot{q}_1 & \dot{q}_1 \\ \dot{q}_2 & \dot{q}_2 & \dot{q}_2 \end{vmatrix} + \frac{\partial s}{\partial \tau}\frac{\partial s}{\partial \alpha}\begin{vmatrix} 1 & 1 & 0 \\ \dot{q}_1 & \dot{q}_1 & \dfrac{\partial q_1}{\partial \beta} \\ \dot{q}_2 & \dot{q}_2 & \dfrac{\partial q_2}{\partial \beta} \end{vmatrix} +$$

$$\frac{\partial s}{\partial \tau}\frac{\partial s}{\partial \beta}\begin{vmatrix} 1 & 0 & 1 \\ \dot{q}_1 & \dfrac{\partial q_1}{\partial \alpha} & \dot{q}_1 \\ \dot{q}_2 & \dfrac{\partial q_2}{\partial \alpha} & \dot{q}_2 \end{vmatrix} + \frac{\partial s}{\partial \tau}\begin{vmatrix} 1 & 0 & 0 \\ \dot{q}_1 & \dfrac{\partial q_1}{\partial \alpha} & \dfrac{\partial q_1}{\partial \beta} \\ \dot{q}_2 & \dfrac{\partial q_2}{\partial \alpha} & \dfrac{\partial q_2}{\partial \beta} \end{vmatrix} = \frac{\partial s}{\partial \tau}\frac{D(q_1, q_2)}{D(\alpha, \beta)} \, .$$

Thus equation (5.14) takes the form

$$\left|\frac{D(x,y,z)}{D(\tau,\alpha,\beta)}\right| = h\frac{\partial s}{\partial \tau}\left|\frac{D(q_1,q_2)}{D(\alpha,\beta)}\right| \, , \quad \frac{\partial s}{\partial \tau} = \frac{1}{\partial \tau/\partial s} = \frac{C}{\sqrt{h^2 + \dot{q}_1^2 + \dot{q}_2^2}} \, . \tag{5.20}$$

Now we have to calculate the determinants in (5.20) on the central ray $\alpha = \alpha_0$, $\beta = \beta_0$, where $q_1 = q_2 = 0$ and $\dot{q}_1 = \dot{q}_2 = 0$, therefore $h = 1$ and

$$\frac{\partial s}{\partial \tau} = C(s, 0, 0) \equiv C_0(s) \quad .$$

Obviously,

$$\left|\frac{D(x,y,z)}{D(\tau,\alpha,\beta)}\right|_{\alpha=\alpha_0,\beta=\beta_0} = C_0 \left|\frac{D(q_1,q_2)}{D(\alpha,\beta)}\right|_{\alpha=\alpha_0,\beta=\beta_0}$$

and the final expression for the amplitude A_0 reads

$$A_0 = \left.\frac{\psi_0(\alpha,\beta)}{\sqrt{(1/C_0)\,|D(q_1,q_2)/D(\alpha,\beta)|}}\right|_{\alpha=\alpha_0,\beta=\beta_0} \, .$$

For the sake of simplicity, let us introduce additional notations

$$\begin{aligned} Q_{1,1} &\equiv \left.\frac{\partial q_1}{\partial \alpha}\right|_{\alpha=\alpha_0,\beta=\beta_0} \; ; \quad Q_{1,2} \equiv \left.\frac{\partial q_1}{\partial \beta}\right|_{\alpha=\alpha_0,\beta=\beta_0} \; ; \\ Q_{2,1} &\equiv \left.\frac{\partial q_2}{\partial \alpha}\right|_{\alpha=\alpha_0,\beta=\beta_0} \; ; \quad Q_{2,2} \equiv \left.\frac{\partial q_2}{\partial \beta}\right|_{\alpha=\alpha_0,\beta=\beta_0} \end{aligned} \tag{5.21}$$

and define

$$\mathbf{Q} = \begin{pmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{pmatrix} \, ,$$

so $\mathbf{Q}$ is the second order matrix. Then the equation for amplitude A_0 can be rewritten in the form

$$A_0 = \frac{\psi_0(\alpha,\beta)}{\sqrt{\dfrac{1}{C_0}|\det \mathbf{Q}|}} \, . \tag{5.22}$$

Note that this is the expression for the amplitude A_0 calculated on the central ray of the ray tube!

Thus, to compute A_0 on a particular ray we have to know the matrix $\mathbf{Q}$ along this ray.

Let us now derive differential equations for the elements of matrix $\mathbf{Q}$.

We start with Euler's equations in Hamiltonian form

$$\frac{d}{ds}q_j = \frac{\partial H}{\partial p_j}\,; \quad \frac{d}{ds}p_j = -\frac{\partial H}{\partial q_j}\,, \quad j = 1, 2. \tag{5.23}$$

Suppose we know the solutions for the rays from the ray tube:

$$q_j = q_j(s, \alpha, \beta)\,, \quad p_j = p_j(s, \alpha, \beta) \quad , j = 1, 2.$$

By inserting them in (5.23) we obtain identities with respect to s, α, β. Let us differentiate (5.23) on the ray parameters α, β and put $\alpha = \alpha_0$, $\beta = \beta_0$, i.e. calculate the derivatives on the central ray. But on the central ray we have $q_1 = q_2 = p_1 = p_2 = 0$, therefore we eventually arrive at the following equations

$$\frac{d}{ds}\left.\frac{\partial q_j}{\partial \alpha}\right|_{\alpha=\alpha_0,\beta=\beta_0} = \sum_{m=1}^{2}\left.\left(\frac{\partial^2 H}{\partial p_j \partial q_m}\frac{\partial q_m}{\partial \alpha} + \frac{\partial^2 H}{\partial p_j \partial p_m}\frac{\partial p_m}{\partial \alpha}\right)\right|_{\alpha=\alpha_0,\beta=\beta_0} \Rightarrow$$

$$\frac{d}{ds}Q_{j,1} = \sum_{m=1}^{2}\left(\left.\frac{\partial^2 H}{\partial p_j \partial q_m}\right|_{q=p=0} Q_{m,1} + \left.\frac{\partial^2 H}{\partial p_j \partial p_m}\right|_{q=p=0} P_{m,1}\right)\,,$$

where we use additional notations

$$P_{1,1} = \left.\frac{\partial p_1}{\partial \alpha}\right|_{\alpha=\alpha_0,\beta=\beta_0}\,; \quad P_{1,2} = \left.\frac{\partial p_1}{\partial \beta}\right|_{\alpha=\alpha_0,\beta=\beta_0}\,;$$

$$P_{2,1} = \left.\frac{\partial p_2}{\partial \alpha}\right|_{\alpha=\alpha_0,\beta=\beta_0}\,; \quad P_{2,2} = \left.\frac{\partial p_2}{\partial \beta}\right|_{\alpha=\alpha_0,\beta=\beta_0}\,.$$

Obviously,

$$\left.\frac{\partial^2 H}{\partial p_j \partial q_m}\right|_{q=p=0} = \left.\frac{\partial^2 H_2}{\partial p_j \partial q_m}\right|_{q=p=0}\,, \quad \left.\frac{\partial^2 H}{\partial p_j \partial p_m}\right|_{q=p=0} = \left.\frac{\partial^2 H_2}{\partial p_j \partial p_m}\right|_{q=p=0}$$

and therefore we can replace H by H_2. Clearly, similar calculations can be carried out for the second part of equations (5.23). Thus, we just arrive at the equations in variations!

Indeed, by taking into account expression for H_2 we obtain, for example,

$$\frac{d}{ds}Q_{j,1} = C_0 P_{j,1}\,.$$

Let us introduce the following second order matrix

$$\mathbf{P} = \begin{pmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{pmatrix}$$

then the desired equations can be presented in a matrix form

$$\frac{d}{ds}\mathbf{Q} = C_0\mathbf{P}\,; \quad \frac{d}{ds}\mathbf{P} = -\frac{1}{C_0^2}\mathbf{C}\mathbf{Q}\,. \tag{5.24}$$

Clearly, they coincide with equations (5.11).

5.6 Point source, initial data for Q and P

In the case of a point source, the initial data for the rays read

$$\vec{r}(\sigma,\alpha,\beta)\big|_{\sigma=0} = \vec{r}_A\,,$$

where $\vec{r}_A$ is the radius vector of the source, and

$$\left.\frac{d\vec{r}(\sigma,\alpha,\beta)}{d\sigma}\right|_{\sigma=0} = \vec{t}^{\,(0)}(\alpha,\beta)\,,$$

where $\vec{t}^{\,(0)}(\alpha,\beta)$ is a unit vector tangent to a ray fixed by the ray parameters α, β. We denote here by σ the arc length along each ray and preserve s as the arc length along the central ray of a ray tube. Then for the central ray field we have

$$\vec{r}(\sigma,\alpha,\beta) = \vec{r}_0(s) + q_1(s,\alpha,\beta)\vec{e}_1(s) + q_2(s,\alpha,\beta)\vec{e}_2(s) \tag{5.25}$$

and σ now has to be considered as a function of s, α, β:

$$d\sigma = \sqrt{h^2 + \dot{q}_1^2 + \dot{q}_2^2}\,ds\,, \quad \sigma = \int_0^s \sqrt{h^2 + \dot{q}_1^2 + \dot{q}_2^2}\,ds\,. \tag{5.26}$$

Due to the fact that all rays start from the same point A, we immediately get that

$$q_1(0,\alpha,\beta) = 0\,, \quad q_2(0,\alpha,\beta) = 0\,, \tag{5.27}$$

and by differentiating the latter equations on α, β we obtain the following initial conditions for the matrix $\mathbf{Q}$

$$\mathbf{Q}\big|_{s=0} = 0\,. \tag{5.28}$$

So, the next step now is to find the initial conditions for the matrix $\mathbf{P}$.

To this end, let us differentiate equation (5.25) with respect to σ and put $\sigma = 0$. (We assume that at the point A we have $\sigma = 0$ and $s = 0$). Note that according to equation (5.26) we can consider s as a function of σ as well, i.e., $s = s(\sigma,\alpha,\beta)$. Thus, now we get consistently

$$\left.\frac{d\vec{r}(\sigma,\alpha,\beta)}{d\sigma}\right|_{\sigma=0} \equiv \vec{t}^{\,(0)}(\alpha,\beta) = \left.\frac{d\vec{r}_0(s)}{ds}\frac{ds}{d\sigma}\right|_{s=0} + \sum_{j=1}^{2}\left.\left(\dot{q}_j\vec{e}_j\frac{ds}{d\sigma} + q_j\frac{d\vec{e}_j}{ds}\frac{ds}{d\sigma}\right)\right|_{s=0}$$

$$= \left[\vec{t}_0 + \sum_{j=1}^{2}\dot{q}_j(0,\alpha,\beta)\vec{e}_j(0)\right]\left.\frac{ds}{d\sigma}\right|_{\sigma=0} \tag{5.29}$$

because $q_j(0,\alpha,\beta) = 0$, $j = 1, 2$.

From equation (5.26) we get

$$\frac{ds}{d\sigma} = \frac{1}{\sqrt{h^2 + \dot{q}_1^2 + \dot{q}_2^2}}$$

and according to the definition of slowness p_j we have

$$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\dot{q}_j}{C\sqrt{h^2 + \dot{q}_1^2 + \dot{q}_2^2}} \qquad \left(L = \frac{\sqrt{h^2 + \dot{q}_1^2 + \dot{q}_2^2}}{C} \right) .$$

By taking it into account, we rewrite equation (5.29) as follows

$$\vec{t}^{\,(0)}(\alpha,\beta) = \vec{t}_0(0)\frac{1}{\sqrt{h^2 + \dot{q}_1^2 + \dot{q}_2^2}} + \sum_{j=1}^{2} C p_j(0,\alpha,\beta)\vec{e}_j(0) \,. \tag{5.30}$$

By multiplying both sides of equation (5.30) by $\vec{e}_1(0)$ and $\vec{e}_2(0)$ we obtain

$$p_j(0,\alpha,\beta) = \frac{1}{C(0,0,0)}(\vec{t}^{\,(0)}(\alpha,\beta), \vec{e}_j(0)) \,, \quad j = 1, 2, \tag{5.31}$$

where $C(0,0,0)$ means the velocity value at the point source.

By differentiating equation (5.31) with respect to the ray parameters α, β and by considering $\alpha = \alpha_0$, $\beta = \beta_0$ we obtain the desired initial data for the elements of the matrix $\mathbf{P}$:

$$P_{j,1}(0) \equiv \left.\frac{\partial p_j(0,\alpha,\beta)}{\partial \alpha}\right|_{\alpha=\alpha_0,\beta=\beta_0} = \frac{1}{C(0,0,0)} \left(\left.\frac{\partial \vec{t}^{\,(0)}}{\partial \alpha}\right|_{\alpha=\alpha_0,\beta=\beta_0}, \vec{e}_j(0) \right) , \tag{5.32}$$

$$P_{j,2}(0) \equiv \left.\frac{\partial p_j(0,\alpha,\beta)}{\partial \beta}\right|_{\alpha=\alpha_0,\beta=\beta_0} = \frac{1}{C(0,0,0)} \left(\left.\frac{\partial \vec{t}^{\,(0)}}{\partial \beta}\right|_{\alpha=\alpha_0,\beta=\beta_0}, \vec{e}_j(0) \right) , j = 1, 2.$$

Let us look at the computational algorithm for the geometrical spreading once again. Suppose, we know the ray connecting a source and an observation point. How can the amplitude A_0 be computed?

The answer can be formulated as follows:

1. We have to construct two unit vectors $\vec{e}_1(s)$ and $\vec{e}_2(s)$ in order to compute derivatives of velocity with respect to q_1 and q_2. To this end, we have to solve a differential equation, e.g., for $\vec{e}_1(s)$, then $\vec{e}_2(s)$ can be found as a vector product of $\vec{e}_1$ and a tangent vector $\vec{t} = d\vec{r}/ds$ to the ray.

2. Next, we have to construct two solutions of the equations in variations (5.11)

$$X^{(1)} = \begin{pmatrix} Q_{1,1} \\ Q_{2,1} \\ P_{1,1} \\ P_{2,1} \end{pmatrix} \quad \text{and} \quad X^{(2)} = \begin{pmatrix} Q_{1,2} \\ Q_{2,2} \\ P_{1,2} \\ P_{2,2} \end{pmatrix}$$

specified by corresponding initial conditions, in the case of a point source by conditions (5.28), (5.32).

Then the absolute value of the determinant of the matrix

$$\mathbf{Q} = \begin{pmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{pmatrix}$$

gives us the geometrical spreading.

3. The final expression for the amplitude is given by formula (5.22)

$$A_0 = \frac{\psi_0(\alpha, \beta)}{\sqrt{\frac{1}{C_0} |\det \mathbf{Q}|}} .$$

5.7 Reduction to a 2D case

1. There is no need to solve any differential equation to find a vector $\vec{e}(s)$ of the ray centered coordinates. It can be chosen as the normal to the ray.

2. Equations in variations now read ($Q_{1,1} \equiv Q$ and $P_{1,1} \equiv P$):

$$\frac{d}{ds} Q = C_0 P \,, \quad \frac{d}{ds} P = -\frac{1}{C_0^2} \left. \frac{\partial^2 C}{\partial q^2} \right|_{q=0} \cdot Q \,,$$

and we have to find one solution of this system specified by a certain initial conditions. For the point source problem we have

$$Q\big|_{s=0} = 0 \,, \quad P\big|_{s=0} = \frac{1}{C(0,0)} \left(\left. \frac{\partial \vec{t}^{\,(0)}}{\partial \alpha} \right|_{\alpha = \alpha_o}, \vec{e}(0) \right) .$$

3. Formula for A_0 takes the form

$$A_0 = \frac{\psi_0(\alpha)}{\sqrt{\frac{1}{C_0} |Q|}} .$$

Example. A homogeneous medium, $C = C_0 = \text{const}$.

Let the ray parameter α be an angle φ between axis x and vector $\vec{t}^{\,(0)}$ tangent to the ray, so that

$$\vec{t}^{\,(0)} = \vec{i} \cos\varphi + \vec{k'} \sin\varphi \,.$$

Suppose that the initial orientation $\vec{e}_{(0)}$ of vector $\vec{e}_{(s)}$ is chosen as follows

$$\vec{e}_{(0)} = -\vec{i} \sin\varphi + \vec{k'} \cos\varphi \,.$$

The equations in variations take the form

$$\frac{d}{ds}Q = C_0 P\,, \quad \frac{d}{ds}P = 0 \Rightarrow \begin{cases} Q(s) = C_0 P(0)s + Q(0) \\ P(s) = P(0) = \text{const} \end{cases} .$$

The initial data in the case of the point source are the following

$$Q|_{s=0} = 0\,, \quad P|_{s=0} = \frac{1}{C_0}\left(\frac{\partial \vec{t}^{\,(0)}}{\partial\varphi}, \vec{e}(0)\right) =$$
$$= \frac{1}{C_0}(-\vec{i}\sin\varphi + \vec{k'}\cos\varphi, -\vec{i}\sin\varphi + \vec{k'}\cos\varphi) = \frac{1}{C_0}\,,$$

therefore $Q(s) = s$ and

$$A_0 = \frac{\psi_0(\alpha)}{\sqrt{\frac{1}{C_0}s}}\,.$$

5.8 An example of a constant gradient velocity model: a point source problem

Assume a velocity $C(z)$ given by the formula

$$C = a + bz\,, \quad a > 0\,, \quad b > 0\,.$$

In this case, the rays are circles with their centers being placed on the straight line $z = z_0$, where velocity is equal to zero, i.e.,

$$C = a + bz_0 = 0 \to z_0 = -\frac{a}{b}\,.$$

Let the central ray of a ray tube be given in the form

$$\begin{cases} z = R\sin\left(\frac{s}{R} + \theta\right) + z_0 \\ x = -R\cos\left(\frac{s}{R} + \theta\right) + x_0 \end{cases},$$

where R and θ are some parameters (see Fig. 5.2) and s is the arc length along the ray. We specify these parameters by the condition in which this ray passes the origin $z = 0$, $x = 0$ when $s = 0$, therefore

$$\begin{cases} 0 = R\sin\theta + z_0 \\ 0 = -R\cos\theta + x_0 \end{cases} \Rightarrow \begin{cases} R = -\frac{z_0}{\sin\theta} \\ x_0 = R\cos\theta = -z_0 \text{ ctg } \theta = \frac{a}{b} \text{ ctg } \theta \end{cases} .$$

We choose vector $\vec{e}_{(0)}$ as follows

$$\vec{e}_{(0)} = -\cos\theta\,\vec{i} + \sin\theta\,\vec{k'}\,.$$

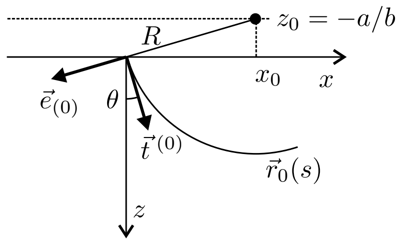

Figure 5.2: Rays in a medium with constant gradient velocity.

The initial tangent vector $\vec{t}^{(0)}(\alpha)$ to a ray will take the form

$$\vec{t}^{(0)}(\alpha) = \sin\alpha\,\vec{i} + \cos\alpha\,\vec{k'}$$

so α has the same geometrical sense as θ and we consider further $\alpha = \theta$, see Fig. 5.2.

Equations for Q and P take the following form

$$\frac{d}{ds}Q = C_0 P\,, \quad \frac{d}{ds}P = 0 \quad \left(\text{due to } \frac{\partial^2 C}{\partial q^2} \equiv 0\right)\,.$$

The general solution reads

$$\begin{aligned} P(s) &= P(0) = \text{const}\,,\\ Q(s) &= P(0)\int_0^s C_0(s)ds + Q(0)\,. \end{aligned}$$

Let us calculate integral $\int_0^s C_0(s)ds$. We have for the central ray

$$\begin{aligned} C_0(s) &= a + b\left[R\sin\left(\frac{s}{R}+\theta\right) + z_0\right] = a + bz_0 + bR\sin\left(\frac{s}{R}+\theta\right) = \\ &= bR\sin\left(\frac{s}{R}+\theta\right)\,, \end{aligned}$$

therefore

$$\int_0^s C_0(s)ds = bR\int_0^s \sin\left(\frac{s}{R}+\theta\right)ds = bR^2\left[\cos\theta - \cos\left(\frac{s}{R}+\theta\right)\right]\,.$$

Initial data for Q and P in the case of the point source are the following

$$\begin{aligned} Q\big|_{s=0} &= 0 \to Q(0) = 0\,,\\ P\big|_{s=0} &= \frac{1}{C(0)}\left(\left.\frac{\partial\vec{t}^{(0)}}{\partial\alpha}\right|_{\alpha=\theta}, \vec{e}(0)\right), C(0) = a\,, \end{aligned}$$

and we have to calculate the scalar product

$$P(0) = \frac{1}{a}(\cos\theta\vec{i} - \sin\theta\vec{k'}, -\cos\theta\vec{i} + \sin\theta\vec{k'}) = \frac{-\cos^2\theta - \sin^2\theta}{a} = -\frac{1}{a}\,.$$

Hence, we get, for our particular solution, the following result

$$P(s) = -\frac{1}{a},$$
$$Q(s) = \frac{-1}{a} bR^2 \left[\cos\theta - \cos\left(\frac{s}{R} + \theta\right)\right] .$$

Thus, for the geometrical spreading J we obtain the following formula

$$J = |Q(s)| = \frac{bR^2}{a} \left|\cos\theta - \cos\left(\frac{s}{R} + \theta\right)\right| .$$

5.9 Solution of the eikonal equation in a vicinity of the central ray

In a vicinity of the central ray $\vec{r}_0(s)$ of a ray tube we use the ray centered coordinates s, q_1, q_2. Let us seek a solution of the eikonal equation as a power series in q_1 and q_2

$$\tau(s, q_1, q_2) = \tau_0(s) + \frac{1}{2} \sum_{i,j=1}^{2} \Gamma_{ij}(s) \cdot q_i q_j + \dots . \tag{5.33}$$

There are no linear terms in q_1 and q_2 because the wavefronts τ = const and the rays are orthogonal. Indeed, for example,

$$\left.\frac{\partial\tau}{\partial q_1}\right|_{q_1=q_2=0}$$

means the derivative along $\vec{e}_1$ which is orthogonal to the central ray. This implies that we differentiate along τ =const and therefore

$$\left.\frac{\partial\tau}{\partial q_1}\right|_{q_1=q_2=0} = 0 .$$

We may say that the coefficients $\Gamma_{ij}, i, j = 1, 2$ form a matrix $\mathbf{\Gamma}$ of the second order and this matrix has to be symmetrical, i.e., $\Gamma_{12} = \Gamma_{21}$ (because $q_1 q_2 = q_2 q_1$).

If we introduce vector $\vec{q}$ by the formula

$$\vec{q} = \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}$$

the second order terms can be presented in the form

$$\frac{1}{2} \sum_{i,j=1}^{2} \Gamma_{ij} q_i q_j = \frac{1}{2} (\mathbf{\Gamma}\vec{q}, \vec{q}) .$$

Now we need to write the eikonal equation in the ray centered coordinates. It is known that

$$\nabla\tau = \frac{\vec{t}}{h}\frac{\partial\tau}{\partial s} + \frac{\partial\tau}{\partial q_1}\vec{e}_1 + \frac{\partial\tau}{\partial q_2}\vec{e}_2$$

and therefore

$$(\nabla\tau, \nabla\tau) = \frac{1}{h^2}\left(\frac{\partial\tau}{\partial s}\right)^2 + \left(\frac{\partial\tau}{\partial q_1}\right)^2 + \left(\frac{\partial\tau}{\partial q_2}\right)^2 = \frac{1}{C^2(s, q_1, q_2)}\,. \tag{5.34}$$

In order to insert expression (5.33) for the eikonal in equation (5.34), we have to decompose $1/h^2$ and $1/C^2$ into power series on q_1 and q_2 also.

Obviously, we have

$$\frac{1}{h} = \frac{1}{1+\sum_{j=1}^{2}\kappa_j q_j} = 1 - \sum_{j=1}^{2}\kappa_j q_j + \left(\sum_{j=1}^{2}\kappa_j q_j\right)^2 + \dots$$

and therefore

$$\frac{1}{h^2} = 1 - 2\sum_{j=1}^{2}\kappa_j q_j + 3\left(\sum_{j=1}^{2}\kappa_j q_j\right)^2 + \dots\,.$$

For $1/C$ we already had the following formula (see section 5.3)

$$\frac{1}{C} = \frac{1}{C_0} - \frac{1}{C_0}\sum_{j=1}^{2}\kappa_j q_j + \frac{1}{C_0}\left(\sum_{j=1}^{2}\kappa_j q_j\right)^2 - \frac{1}{2C_0^2}\sum_{i,j=1}^{2}C_{ij}q_i q_j + \dots\,.$$

It follows from here that

$$\frac{1}{C^2} = \frac{1}{C_0^2} - \frac{2}{C_0^2}\sum_{j=1}^{2}\kappa_j q_j + \frac{3}{C_0^2}\left(\sum_{j=1}^{2}\kappa_j q_j\right)^2 - \frac{1}{C_0^3}\sum_{i,j=1}^{2}C_{ij}q_i q_j + \dots\,.$$

By taking into account equation (5.33), we obtain

$$\frac{\partial\tau}{\partial s} = \tau_0' + \frac{1}{2}\sum_{i,j=1}^{2}\Gamma_{ij}' q_i q_j + \cdots\,; \left(\frac{\partial\tau}{\partial s}\right)^2 = (\tau_0')^2 + \tau_0'\sum_{i,j=1}^{2}\Gamma_{ij}' q_i q_j + \dots\,;$$

where by $'$ we denote derivative d/ds, and

$$\left(\frac{\partial\tau}{\partial q_1}\right)^2 = \left(\Gamma_{11}q_1 + \Gamma_{12}q_2\right)^2, \quad \left(\frac{\partial\tau}{\partial q_2}\right)^2 = \left(\Gamma_{12}q_1 + \Gamma_{22}q_2\right)^2.$$

We get the following result from the two last equations

$$\left(\frac{\partial\tau}{\partial q_1}\right)^2 + \left(\frac{\partial\tau}{\partial q_2}\right)^2 = (\Gamma_{11}^2 + \Gamma_{12}^2)q_1^2 + 2(\Gamma_{11}\Gamma_{12} + \Gamma_{12}\Gamma_{22})q_1 q_2 + (\Gamma_{12}^2 + \Gamma_{22}^2)q_2^2 =$$

$$= \sum_{i,j=1}^{2}(\mathbf{\Gamma}^2)_{ij}q_i q_j$$

because

$$\mathbf{\Gamma}^2 = \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12} & \Gamma_{22} \end{pmatrix} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12} & \Gamma_{22} \end{pmatrix} = \begin{pmatrix} \Gamma_{11}^2 + \Gamma_{12}^2 & \Gamma_{11}\Gamma_{12} + \Gamma_{12}\Gamma_{22} \\ \Gamma_{11}\Gamma_{12} + \Gamma_{12}\Gamma_{22} & \Gamma_{12}^2 + \Gamma_{22}^2 \end{pmatrix} .$$

Now we are able to insert all these expressions in the eikonal equation (5.34) and to equate those terms which contain the same power of q_1 and q_2 in both sides of the eikonal equation.

For the terms, containing q_1^0 and q_2^0 we easily obtain

$$(\tau_0^{'})^2 = \frac{1}{C_0^2} \to \tau_0' = \pm \frac{1}{C_0}$$

and for the wave propagating along the central ray we have to take +, i.e.,

$$\tau_0' = \frac{1}{C_0} \to \tau_0 = \int_{s_0}^{s} \frac{ds}{C_0} + \tau_0(s_0) \, .$$

Linear terms in q_1 and q_2 give us the following equality

$$-2(\tau_0')^2 \sum_{j=1}^{2} \kappa_j q_j = -\frac{2}{C_0^2} \sum_{j=1}^{2} \kappa_j q_j$$

which is already fulfilled due to equation for τ_o'.

Second order terms lead to the following equality

$$3(\tau_0')^2 \left(\sum_{j=1}^{2} \kappa_j q_j \right)^2 + \tau_0' \sum_{i,j=1}^{2} \Gamma_{ij}' q_j q_j + \sum_{i,j=1}^{2} (\mathbf{\Gamma}^2)_{ij} q_i q_j =$$

$$= \frac{3}{C_0^2} \left(\sum_{j=1}^{2} \kappa_j q_j \right)^2 - \frac{1}{C_0^3} \sum_{i,j=1}^{2} C_{ij} q_i q_j \, .$$

This equation can be presented as follows

$$\sum_{i,j=1}^{2} q_i q_j \left\{ \tau_0' \Gamma_{ij}' + (\Gamma^2)_{ij} + \frac{1}{C_0^3} C_{ij} \right\} = 0$$

and to satisfy it we have to impose

$$\frac{1}{C_0} \frac{d}{ds} \Gamma_{ij} + (\mathbf{\Gamma}^2)_{ij} + \frac{1}{C_0^3} C_{ij} = 0 \, , \quad \text{for } i, j = 1, 2.$$

This is a system of ordinary but nonlinear differential equations because of the term $\mathbf{\Gamma}^2$. In matrix form it can be written as follows

$$\frac{d}{ds} \mathbf{\Gamma} + C_0 \mathbf{\Gamma}^2 + \frac{1}{C_0^2} \mathbf{C} = 0 \, . \tag{5.35}$$

This is called Ricatti's matrix equation. How do we integrate this equation?

It can be reduced to a system of linear equations by means of the following substitution. Let us seek matrix $\mathbf{\Gamma}$ in the form

$$\mathbf{\Gamma} = \mathbf{P}\mathbf{Q}^{-1} \tag{5.36}$$

where matrices of the second order $\mathbf{P}$ and $\mathbf{Q}$ are so far unknown.

Further, we have obviously

$$\frac{d}{ds}\mathbf{\Gamma} = \frac{d\mathbf{P}}{ds}\mathbf{Q}^{-1} + \mathbf{P}\frac{d}{ds}\mathbf{Q}^{-1}\,.$$

In order to find the derivative of $\mathbf{Q}^{-1}$ we can differentiate the identity $\mathbf{Q}\mathbf{Q}^{-1} = \mathbf{E}$. Indeed,

$$\frac{d}{ds}(\mathbf{Q}\mathbf{Q}^{-1}) = \frac{d}{ds}\mathbf{E} = 0$$

but

$$\frac{d}{ds}(\mathbf{Q}\mathbf{Q}^{-1}) = \frac{d\mathbf{Q}}{ds}\mathbf{Q}^{-1} + \mathbf{Q}\frac{d\mathbf{Q}^{-1}}{ds}$$

and therefore

$$\frac{d\mathbf{Q}^{-1}}{ds} = -\mathbf{Q}^{-1}\frac{d\mathbf{Q}}{ds}\mathbf{Q}^{-1}\,.$$

Now we can substitute matrix $\mathbf{\Gamma}$ in Ricatti's matrix equation (5.35) by the left hand side of equation (5.36)

$$\frac{d\mathbf{P}}{ds}\mathbf{Q}^{-1} - \mathbf{P}\mathbf{Q}^{-1}\frac{d\mathbf{Q}}{ds}\mathbf{Q}^{-1} + C_0\mathbf{P}\mathbf{Q}^{-1}\mathbf{P}\mathbf{Q}^{-1} + \frac{1}{C_0^2}\mathbf{C} =$$
$$= \frac{d\mathbf{P}}{ds}\mathbf{Q}^{-1} + \frac{1}{C_0^2}\mathbf{C} - \mathbf{P}\mathbf{Q}^{-1}\left(\frac{d\mathbf{Q}}{ds} - C_0\mathbf{P}\right)\mathbf{Q}^{-1} = 0$$

and in order to satisfy the latter equation we may simply put

$$\frac{d\mathbf{Q}}{ds} = C_0\mathbf{P}\,, \quad \frac{d\mathbf{P}}{ds} = -\frac{1}{C_0^2}\mathbf{C}\mathbf{Q}\,. \tag{5.37}$$

But now we recognize that equations (5.37) coincide with the equations in variations (see formulas (5.24)).

So, to solve Ricatti's equation (5.35) for matrix $\mathbf{\Gamma}$ we may use some solutions of the equations in variations, i.e., the linearized equations for the rays from a ray tube around the central ray!

But an important question arises on the way. Will matrix $\mathbf{\Gamma}$ be symmetrical?

In order to get the answer to the question, we have to check the following equality

$$\mathbf{\Gamma}^T - \mathbf{\Gamma} = 0 \to (\mathbf{P}\mathbf{Q}^{-1})^T - \mathbf{P}\mathbf{Q}^{-1} = 0 \to (\mathbf{Q}^T)^{-1}\mathbf{P}^T - \mathbf{P}\mathbf{Q}^{-1} = 0\,.$$

Let us multiply the latter equation by $\mathbf{Q}^T$ from the left side and by $\mathbf{Q}$ from the right side, then we get the following equation

$$\mathbf{P}^T\mathbf{Q} - \mathbf{Q}^T\mathbf{P} = 0\,,$$

which is equivalent to the initial one (if $\det\mathbf{Q} \neq 0$). By substituting here the matrices $\mathbf{P}$ and $\mathbf{Q}$ we obtain

$$\begin{aligned}\mathbf{P}^T\mathbf{Q} - \mathbf{Q}^T\mathbf{P} &= \begin{pmatrix} P_{1,1} & P_{2,1} \\ P_{1,2} & P_{2,2} \end{pmatrix}\begin{pmatrix} Q_{1,1} & Q_{1,2} \\ Q_{2,1} & Q_{2,2} \end{pmatrix} - \\ &- \begin{pmatrix} Q_{1,1} & Q_{2,1} \\ Q_{1,2} & Q_{2,2} \end{pmatrix}\begin{pmatrix} P_{1,1} & P_{1,2} \\ P_{2,1} & P_{2,2} \end{pmatrix} = \\ &= \begin{pmatrix} 0 & +(JX^{(1)}, X^{(2)}) \\ -(JX^{(1)}, X^{(2)}) & 0 \end{pmatrix}\end{aligned}$$

where by $X^{(1)}$ and $X^{(2)}$ we denote the following vector-columns

$$X^{(1)} = \begin{pmatrix} Q_{1,1} \\ Q_{2,1} \\ P_{1,1} \\ P_{2,1} \end{pmatrix} \quad \text{and } X^{(2)} = \begin{pmatrix} Q_{1,2} \\ Q_{2,2} \\ P_{1,2} \\ P_{2,2} \end{pmatrix}.$$

Thus we obtain the following result: if the $\mathbf{J}$-scalar product of two solutions $X^{(1)}$ and $X^{(2)}$ of the equations in variations is equal to zero, the matrix $\mathbf{\Gamma}$ will be symmetrical.

Remark 1. Notice that the matrix $\mathbf{Q}$ cannot be singular, i.e., $\det\mathbf{Q} \neq 0$ at least identically with respect to s. Otherwise the matrix $\mathbf{\Gamma}$ will not exist. However, precisely as regards to caustics, $\det\mathbf{Q} = 0$ and therefore the elements of matrix $\mathbf{\Gamma}$ become singular on caustics.

Remark 2. For a point source we had $Q_{1,1} = Q_{2,1} = Q_{1,2} = Q_{2,2} = 0$ at the initial point $s = 0$, therefore $(JX^{(1)}, X^{(2)}) = \big(JX^{(1)}(0), X^{(2)}(0)\big) = 0$ and $\mathbf{\Gamma}^T = \mathbf{\Gamma}$.

5.10 Reduction to a 2D case

In this case we can simply assume $q_2 \equiv 0$ and put $q_1 \equiv q$, therefore for the eikonal τ we have

$$\tau(s, q) = \tau_0(s) + \frac{1}{2}\Gamma q^2 + \cdots\,, \tag{5.38}$$

where Γ now is not a matrix but a scalar function. It must satisfy Ricatti's equation

$$\Gamma' + C_0\Gamma^2 + \frac{1}{C_0^2}\left.\frac{\partial^2 C}{\partial q^2}\right|_{q=0} = 0\,.$$

Now we simply make

$$\Gamma = \frac{P}{Q}$$

and obtain the equations in variations

$$Q' = C_0 P \,, \quad P' = -\frac{1}{C_0^2}\frac{\partial^2 C}{\partial q^2}\Big|_{q=0} \cdot Q \,. \tag{5.39}$$

Example 1. A homogeneous medium, point source.
Initial data for Q and P are the following

$$Q(0) = 0 \,, \quad P(0) = \frac{1}{C_0} \quad .$$

The corresponding solution of (5.39) reads

$$Q(s) = s \,, \quad P(s) = \frac{1}{C_0} \quad ,$$

therefore for the eikonal (5.38) we get

$$\tau(s,q) = \frac{s}{C_0} + \frac{1}{2}\frac{q^2}{C_0 s} + \cdots \,.$$

Consider now wave fronts $\tau =$ const in a vicinity of the central ray. Make const $= \tau_*$ and let $s = s_*$, $q = 0$ be the point of intersection between the central ray and the wave front. So $\tau_* = s_*/C_0$ and we get

$$\tau(s,q) = \frac{s}{C_0} + \frac{1}{2}\frac{q^2}{C_0 s} + \cdots = \frac{s_*}{C_0} \,.$$

It follows from here that approximately

$$s - s_* \cong -\frac{1}{2}\frac{q^2}{s_*} \,,$$

and therefore

$$C_0 \left.\frac{P}{Q}\right|_{s=s_*} = \frac{1}{s_*} = \frac{1}{R_*} \,,$$

where R_* is the radius of curvature of the wave front on the central ray at the point $s = s_*$. This example illustrates the geometrical sense of $C_0\mathbf{\Gamma}$ as the curvature of the wave front at a point of intersection with the central ray of a ray tube. It also holds true in 3D, but in this case the matrix $C_0\mathbf{\Gamma}$ describes the main curvatures of the wave front and their orientation with respect to vectors $\vec{e}_1$ and $\vec{e}_2$.

Example 2. Constant gradient velocity model, point source problem.
As above we assume velocity C to be given in the form $C = a + bz$; $a, b > 0$.
In this case we had

$$P(s) = -\frac{1}{a} \,, \quad Q(s) = \frac{-1}{a} bR^2 \left[\cos\theta - \cos\left(\frac{s}{R} + \theta\right)\right] \,,$$

hence, for the eikonal we obtain now

$$\tau(s,q) = \tau_0(s) + \frac{1}{2}\frac{q^2}{bR^2\left[\cos\theta - \cos\left(\frac{s}{R} + \theta\right)\right]} + \cdots \,,$$

where

$$\begin{aligned}\tau_0(s) &= \int_0^s \frac{ds}{C_0(s)} = \frac{1}{bR}\int_0^s \frac{ds}{\sin(\frac{s}{R}+\theta)} = \frac{1}{b}\ln\mathrm{tg}\left(\frac{s}{2R}+\frac{\theta}{2}\right)\Bigg|_{s=0}^{s=s} = \\ &= \frac{1}{b}\left[\ln\mathrm{tg}\left(\frac{s}{2R}+\frac{\theta}{2}\right) - \ln\mathrm{tg}\,\frac{\theta}{2}\right] = \frac{1}{b}\ln\frac{\mathrm{tg}\left(\frac{s}{2R}+\frac{\theta}{2}\right)}{\mathrm{tg}\,\frac{\theta}{2}}\,.\end{aligned}$$

5.11 Reduction to 2.5D

In this case velocity C does not depend on q_2 and matrix $\mathbf{C}$ takes the form

$$\mathbf{C} = \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix}, \quad C_{11} = \frac{\partial^2 C}{\partial q_1^2}\Big|_{q_1=0}\,.$$

The equations in variations in matrix form read

$$\begin{aligned}\begin{pmatrix} \dot{Q}_{1,1} & \dot{Q}_{1,2} \\ \dot{Q}_{2,1} & \dot{Q}_{2,2} \end{pmatrix} &= \begin{pmatrix} C_0P_{1,1} & C_0P_{1,2} \\ C_0P_{2,1} & C_0P_{2,2} \end{pmatrix}; \\ \begin{pmatrix} \dot{P}_{1,1} & \dot{P}_{1,2} \\ \dot{P}_{2,1} & \dot{P}_{2,2} \end{pmatrix} &= \begin{pmatrix} -\frac{C_{11}}{C_0^2}Q_{1,1} & -\frac{C_{11}}{C_0^2}Q_{1,2} \\ 0 & 0 \end{pmatrix}\end{aligned}$$

and then for the elements of the matrices we get

$$\dot{Q}_{1,j} = C_0P_{1,j}\,; \quad \dot{P}_{1,j} = -\frac{C_{11}}{C_0^2}Q_{1,j}\,, \quad j=1,2, \tag{5.40}$$

$$\dot{Q}_{2,j} = C_0P_{2,j}\,; \quad \dot{P}_{2,j} = 0\,, \qquad j=1,2. \tag{5.41}$$

Equations (5.40) and (5.41) are independent systems of ordinary differential equations. The latter one can be integrated immediately and its general solution reads

$$P_{2,j}(s) = P_{2,j}(0) = \mathrm{const}, \quad Q_{2,j}(s) = P_{2,j}(0)\int_0^s C_0(s)ds + Q_{2,j}(0)\,.$$

Now let us consider the initial data in detail. It is convenient to use the spherical angles θ and φ as the ray parameters. For the unit vector $\vec{t}^{\,(0)}$ tangent to the ray we have in this case

$$\vec{t}^{\,(0)}(\theta,\varphi) = \sin\theta\cos\varphi\,\vec{i} + \sin\theta\sin\varphi\,\vec{j} + \cos\theta\,\vec{k}\,.$$

Then, let us choose an initial orientation of the unit vectors $\vec{e}_1(0)$ and $\vec{e}_2(0)$ as follows

$$\vec{e}_1(0) = \frac{\partial\vec{t}^{\,(0)}}{\partial\theta}\Big|_{\varphi=\varphi_0,\theta=\theta_0}, \quad \vec{e}_2(0) = \frac{1}{\sin\theta_0}\,\frac{\partial\vec{t}^{\,(0)}}{\partial\varphi}\Big|_{\varphi=\varphi_0,\theta=\theta_0},$$

where φ_0, θ_0 correspond to the central ray of a ray tube.

Further, we shall consider the rays from x, z plane only and assume that the velocity does not depend on y. This is precisely what provides simplification.

Make then $\varphi_0 = 0$ and for the central ray of a ray tube we get

$$\vec{t}(\theta_0, 0) = \sin\theta_0\,\vec{i} + \cos\theta_0\,\vec{k'}\ .$$

For the unit vectors $\vec{e}_1(0)$ and $\vec{e}_2(0)$ we obtain respectively

$$\begin{aligned}
\vec{e}_1(0) &= \left.\frac{\partial \vec{t}^{\,(0)}}{\partial\theta}\right|_{\theta=\theta_0,\varphi=0} = \cos\theta_0\,\vec{i} - \sin\theta_0\,\vec{k'}\ ,\\
\vec{e}_2(0) &= \left.\frac{\partial \vec{t}^{\,(0)}}{\partial\varphi}\right|_{\theta=\theta_0,\varphi=0} \cdot \frac{1}{\sin\theta_0} = \vec{j}\ .
\end{aligned}$$

The initial data for $Q_{i,j}$ in case of a point source are the following

$$Q_{i,j}(0) = 0\ , \quad i,j = 1,2\ .$$

In order to obtain the initial data for $P_{i,j}$ we have to carry out some calculations.

We had the following expressions for slownesses p_j

$$p_j(0,\theta,\varphi) = \frac{1}{C(0,0,0)}(\vec{t}^{\,(0)}(\theta,\varphi), \vec{e}_j(0))\ , \quad j = 1,2,$$

where $\alpha = \theta$ and $\beta = \varphi$ are considered. By differentiating the latter formulas with respect to θ and φ we obtain

$$\begin{aligned}
P_{j,1}(0) &= \left.\frac{\partial p_j}{\partial\theta}\right|_{\theta=\theta_0,\varphi=0} = \frac{1}{C(0,0,0)}\left.\left(\frac{\partial\vec{t}^{\,(0)}}{\partial\theta}, \vec{e}_j(0)\right)\right|_{\theta=\theta_0,\varphi=0} =\\
&= \frac{1}{C(0,0,0)}(\vec{e}_1(0), \vec{e}_j(0))\ ,\\
P_{j,2}(0) &= \left.\frac{\partial p_j}{\partial\varphi}\right|_{\theta=\theta_0,\varphi=0} = \frac{1}{C(0,0,0)}\left.\left(\frac{\partial\vec{t}^{\,(0)}}{\partial\varphi}, \vec{e}_j(0)\right)\right|_{\theta=\theta_0,\varphi=0} =\\
&= \frac{\sin\theta_0}{C(0,0,0)}(\vec{e}_2(0), \vec{e}_j(0))\ .
\end{aligned}$$

Now we get the final results for column-vectors $X^{(j)}(0)$, $j = 1, 2$,

$$
\begin{aligned}
X^{(1)}(0) &= \begin{pmatrix} Q_{1,1}(0) \\ Q_{2,1}(0) \\ P_{1,1}(0) \\ P_{2,1}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{1}{C(0,0,0)} \\ 0 \end{pmatrix} ; \\
X^{(2)}(0) &= \begin{pmatrix} Q_{1,2}(0) \\ Q_{2,2}(0) \\ P_{1,2}(0) \\ P_{2,2}(0) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \dfrac{\sin\theta_0}{C(0,0,0)} \end{pmatrix} .
\end{aligned}
\tag{5.42}
$$

Note, that $(\mathbf{J}X^{(1)}(0), X^{(2)}(0)) = 0$ and matrix $\mathbf{\Gamma}$ will be symmetrical!

Based on initial data (5.42) we can present $X^{(1)}$ and $X^{(2)}$ for arbitrary s in the following form

$$
X^{(1)}(s) = \begin{pmatrix} Q_{1,1}(s) \\ 0 \\ P_{1,1}(s) \\ 0 \end{pmatrix} , \quad X^{(2)}(s) = \begin{pmatrix} 0 \\ Q_{2,2}(s) \\ 0 \\ P_{2,2}(s) \end{pmatrix}
$$

where

$$
P_{2,2}(s) = P_{2,2}(0) = \frac{\sin\theta_0}{C(0,0,0)}
$$

and

$$
Q_{2,2}(s) = P_{2,2}(0) \int_0^s C_0 ds = \frac{\sin\theta_0}{C(0,0,0)} \int_0^s C_0(s) ds .
$$

As for $Q_{1,1}(s)$ and $P_{1,1}(s)$ we only know that they satisfy equations (5.40) and do not equal to zero identically with respect to s.

These results follow from the systems of equations (5.40),(5.41) and from initial conditions (5.42) almost straightforward. Indeed, both systems of equations (5.40) and (5.41) are homogeneous, so the unique solution of such a system, specified also by homogeneous initial conditions, is equal to zero identically.

Thus, we obtain the following structure for matrixes $\mathbf{Q}$ and $\mathbf{P}$

$$
\begin{aligned}
\mathbf{Q}(s) &= \begin{pmatrix} Q_{1,1}(s) & 0 \\ 0 & Q_{2,2}(s) \end{pmatrix} , \quad \Rightarrow \mathbf{Q}^{-1}(s) = \begin{pmatrix} Q_{1,1}^{-1}(s) & 0 \\ 0 & Q_{2,2}^{-1}(s) \end{pmatrix} , \\
\hat{\mathbf{P}}(s) &= \begin{pmatrix} P_{1,1}(s) & 0 \\ 0 & P_{2,2}(s) \end{pmatrix} .
\end{aligned}
$$

Hence, we get for matrix $\mathbf{\Gamma} = \mathbf{P}\mathbf{Q}^{-1}$ the following expression

$$
\mathbf{\Gamma} = \begin{pmatrix} \dfrac{P_{1,1}(s)}{Q_{1,1}(s)} & 0 \\ 0 & \dfrac{P_{2,2}(s)}{Q_{2,2}(s)} \end{pmatrix} .
$$

Eventually, the eikonal τ in the 2.5D case takes the form

$$\tau(s, q_1, q_2) = \int_0^s \frac{ds}{C_0(s)} + \frac{1}{2}\frac{P_{1,1}(s)}{Q_{1,1}(s)} q_1^2 + \frac{1}{2}\frac{P_{2,2}(s)}{Q_{2,2}(s)} q_2^2 + \cdots .$$

Example. Constant gradient velocity model, a point source, 2.5D case.
We assume that $C = a + bz$. The equations (5.40) and (5.41) take the form

$$\begin{aligned} \dot{Q}_{1,j} &= C_0 P_{1,j} \quad , \dot{P}_{1,j} = 0\,, \quad j = 1, 2\,, \\ \dot{Q}_{2,j} &= C_0 P_{2,j} \quad , \dot{P}_{2,j} = 0\,, \quad j = 1, 2\,. \end{aligned}$$

Now we have to take into account initial conditions (5.42).
For $Q_{1,1}$ and $P_{1,1}$ we obtain

$$P_{1,1}(0) = \frac{1}{C(0,0,0)} = \frac{1}{a}\,, \quad Q_{1,1}(0) = 0\,,$$

and therefore

$$\begin{aligned} P_{1,1}(s) &= \frac{1}{a}\,; \quad Q_{1,1}(s) = \frac{1}{a}\int_0^s C_0(s) ds = \frac{bR}{a}\int_0^s \sin(\frac{s}{R} + \theta) ds = \\ &= \frac{bR^2}{a}\left[\cos\theta - \cos(\frac{s}{R} + \theta)\right] . \end{aligned}$$

For $Q_{2,2}$ and $P_{2,2}$ we obtain

$$P_{2,2}(0) = \frac{\sin\theta_0}{C(0,0,0)} = \frac{\sin\theta_0}{a}\,; \quad Q_{2,2}(0) = 0$$

therefore

$$\begin{aligned} P_{2,2}(s) &= \frac{\sin\theta_0}{a}\,, \quad Q_{2,2}(s) = P_{2,2}(0)\int_0^s C_0(s) ds = \\ &= \frac{\sin\theta_0}{a} bR^2 \left[(\cos\theta - \cos(\frac{s}{R} + \theta)\right] . \end{aligned}$$

Finally, we get the following result for the eikonal τ:

$$\tau(s, q_1, q_2) = \tau_0(s) + \frac{q_1^2 + q_2^2}{2bR^2\left[\cos\theta - \cos(\frac{s}{R} + \theta)\right]} + \cdots .$$

5.12 Solution of the transport equation in a vicinity of the central ray

The transport equation for the main term of amplitude A_0 reads

$$2(\nabla A_0, \nabla\tau) + A_0 \Delta\tau = 0\,. \tag{5.43}$$

We seek a solution of this equation in the form of a power series on q_1 and q_2

$$A_0(s,q_1,q_2) = A_0^{(0)}(s) + \sum_{j=1}^{2} A_0^{(j)}(s) q_j + \cdots \,. \tag{5.44}$$

But this time we shall look only for $A_0^{(0)}(s)$ which is precisely the amplitude A_0 computed on the central ray of a ray tube.

In the ray centered coordinates s, q_1, q_2 we have

$$\nabla = \frac{\vec{t}}{h}\frac{\partial}{\partial s} + \vec{e}_1 \frac{\partial}{\partial q_1} + \vec{e}_2 \frac{\partial}{\partial q_2} \,,$$

and for the Laplace operator

$$\Delta\tau = \frac{1}{h}\left\{ \frac{\partial}{\partial s}\left(\frac{1}{h}\frac{\partial \tau}{\partial s}\right) + \frac{\partial}{\partial q_1}\left(h\frac{\partial \tau}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(h\frac{\partial \tau}{\partial q_2}\right)\right\} =$$

$$\frac{1}{h^2}\frac{\partial^2 \tau}{\partial s^2} - \frac{1}{h^3}\frac{\partial h}{\partial s}\frac{\partial \tau}{\partial s} + \frac{\partial^2 \tau}{\partial q_1^2} + \frac{\partial^2 \tau}{\partial q_2^2} + \frac{1}{h}\frac{\partial h}{\partial q_1}\frac{\partial \tau}{\partial q_1} + \frac{1}{h}\frac{\partial h}{\partial q_2}\frac{\partial \tau}{\partial q_2} \,.$$

For the eikonal τ we may use the following formula

$$\tau(s,q_1,q_2) = \tau_0(s) + \frac{1}{2}\sum_{i,j=1}^{2} \Gamma_{ij}(s) q_i q_j + \cdots \,. \tag{5.45}$$

Now we have to insert formulas (5.44), (5.45) into the transport equation (5.43) and to gather the terms of the same order with respect to q_1 and q_2. But for the main term $A_0^{(0)}(s)$, it is enough to calculate the left-hand side of (5.43) precisely on the central ray. Let us conduct this program step by step.

Evidently,

$$(\nabla A_0, \nabla \tau)\big|_{q_1=q_2=0} = \left(\vec{t}\,\frac{d\tau_0}{ds}, \vec{t}\,\frac{dA_0^{(0)}}{ds}\right) = \frac{dA_0^{(0)}}{ds}\frac{d\tau_0}{ds} = \frac{1}{C_0}\frac{dA_0^{(0)}}{ds}$$

and

$$\Delta\tau\big|_{q_1=q_2=0} = \left(\frac{\partial^2 \tau}{\partial s^2} + \frac{\partial^2 \tau}{\partial q_1^2} + \frac{\partial^2 \tau}{\partial q_2^2} + \kappa_1 \frac{\partial \tau}{\partial q_1} + \kappa_2 \frac{\partial \tau}{\partial q_2}\right)\bigg|_{q_1=q_2=0} =$$

$$\frac{d^2\tau_0}{ds^2} + \operatorname{tr}\mathbf{\Gamma} = -\frac{1}{C_0^2}\frac{dC_0}{ds} + \operatorname{tr}\mathbf{\Gamma} \quad ,$$

where $\operatorname{tr}\mathbf{\Gamma} = \Gamma_{11} + \Gamma_{22}$.

Hence, it follows now from equation (5.43) that

$$\frac{2}{C_0}\frac{dA_0^{(0)}}{ds} + A_0^{(0)}\left(-\frac{1}{C_0^2}\frac{dC_0}{ds} + \operatorname{tr}\mathbf{\Gamma}\right) = 0$$

and we can develop the latter equation as follows

$$
\begin{aligned}
&\frac{1}{A_0^{(0)}}\frac{dA_0^{(0)}}{ds} = +\frac{1}{2}\frac{1}{C_0}\frac{dC_0}{ds} - \frac{C_0}{2}\operatorname{tr}\mathbf{\Gamma} \Rightarrow \\
&\frac{d}{ds}\ln A_0^{(0)} = \frac{1}{2}\frac{d}{ds}\ln C_0 - \frac{1}{2}C_0\operatorname{tr}\mathbf{\Gamma} \Rightarrow \\
&\ln A_0^{(0)} = \ln\sqrt{C_0} - \frac{1}{2}\int_{s_0}^{s} C_0(s)\operatorname{tr}\mathbf{\Gamma}(s)ds + \ln\psi_0\ ,
\end{aligned}
$$

where by $\ln\psi_0$ we denote a constant of integration.

Eventually,

$$
A_0^{(0)} = \psi_0\sqrt{C_0}\exp\left\{-\frac{1}{2}\int_{s_0}^{s} C_0\operatorname{tr}\mathbf{\Gamma}ds\right\}\ . \tag{5.46}
$$

Next we have to calculate the integral in equation (5.46). To this end, consider the equations in variations in matrix form

$$
\frac{d}{ds}\mathbf{Q} = C_0\mathbf{P}\ , \quad \frac{d}{ds}\mathbf{P} = -\frac{1}{C_0^2}\mathbf{C}\mathbf{Q}\,. \tag{5.47}
$$

The first one can be rewritten as follows

$$
\frac{d}{ds}\mathbf{Q} = C_0\mathbf{P}\mathbf{Q}^{-1}\mathbf{Q} = C_0\mathbf{\Gamma}\mathbf{Q}\ , \quad \mathbf{\Gamma} = \mathbf{P}\mathbf{Q}^{-1}\,. \tag{5.48}
$$

Suppose, matrix $\mathbf{Q}$ satisfies equation (5.48). Let us calculate the derivative on s of the determinant of $\mathbf{Q}$ and insert the result in equation (5.48)

$$
\begin{aligned}
\frac{d}{ds}\det\mathbf{Q} &= \frac{d}{ds}(Q_{1,1}Q_{2,2} - Q_{1,2}Q_{2,1}) = \\
&= C_0(\Gamma_{11}Q_{1,1} + \Gamma_{12}Q_{2,1})Q_{2,2} - C_0(\Gamma_{11}Q_{1,2} + \Gamma_{12}Q_{2,2})Q_{2,1} + \\
&\quad + C_0Q_{1,1}(\Gamma_{21}Q_{1,2} + \Gamma_{22}Q_{2,2}) - C_0Q_{1,2}(\Gamma_{21}Q_{1,1} + \Gamma_{22}Q_{2,1}) = \\
&= C_0\operatorname{tr}\mathbf{\Gamma}\ \det\mathbf{Q}\ ,
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\frac{1}{\det\mathbf{Q}}\frac{d}{ds}\det\mathbf{Q} &= C_0\operatorname{tr}\mathbf{\Gamma} \Rightarrow d\ln\det Q = C_0\operatorname{tr}\Gamma\ ds \Rightarrow \\
\ln\det\mathbf{Q} &= \int_{s_0}^{s} C_0\operatorname{tr}\mathbf{\Gamma}ds + \text{const.}
\end{aligned}
$$

By inserting the latter formula into equation (5.46) we obtain the final result

$$
A_0^{(0)} = \frac{\psi_0}{\sqrt{\dfrac{1}{C_0}|\det\mathbf{Q}|}}\ ,
$$

which we have already established.

Remark. The latter formula proves that $\det\mathbf{Q} = 0$ on caustics and only on caustics where the geometrical spreading vanishes.

5.13 How can the initial data for the amplitude be found?

A problem with the initial amplitudes in the ray theory arises in the case of point sources because the geometrical spreading for the central ray field vanishes at the source point (it is a caustic!). But point sources are in broad use in theoretical and applied geophysics. The problem has been solved by means of matching of asymptotics. The basic ideas of this method can be explained as follows.

In some small vicinity of the point source under consideration we can develop a perturbation method for the equations of motion (in our case the reduced wave equation) which provides asymptotics to the problem in closed analytical form. Then we have to derive the long distance asymptotics (in terms of the wavelength) from those analytical formulas and that asymptotics has normally the form of a ray series. On the other hand, we can construct a ray method series for the wave field and then simplify it in that small vicinity of the point source because the rays can be regarded there as perturbed straight lines. Finally, by comparing both these asymptotics, the desired formulas for the initial amplitudes appear. We demonstrate this procedure below for the main amplitude of the ray series. For more details see Babich and Kirpichnikova (1979).

Consider first the point source problem in an inhomogeneous medium, 2D.

In this case we have to consider the central ray field with the center at the source and to construct the eikonal τ and amplitude A_0.

Eventually, we obtain the following formula

$$U = A_0 e^{i\omega\tau} = \frac{\psi_0(\alpha)}{\sqrt{\frac{1}{C}J}} e^{i\omega\tau} \,. \tag{5.49}$$

In some small vicinity of the source, we may assume velocity C to be constant and equal to its value at the source $C = C(0,0) \equiv C(0)$, (we assume that the source is located at the origin of Cartesian coordinates). But in such a case we arrive at the corresponding problem for homogeneous medium and therefore

$$\tau = \int_0^s \frac{ds}{C} \simeq \frac{s}{C(0)} \,; \quad J = |Q| \simeq s \,,$$

where s is the arc length along the rays. Evidently, $s \simeq \sqrt{x^2+z^2}$.

If we denote $r = \sqrt{x^2+z^2}$, then in this vicinity of the source we get

$$U \simeq \frac{\psi_0(\alpha)}{\sqrt{\frac{1}{C(0)}r}} \exp\left\{ i\frac{\omega}{C(0)}r \right\} \,. \tag{5.50}$$

On the other hand, corresponding mathematical formulation of the point source problem is the following

$$\left(\Delta + \frac{\omega^2}{C^2}\right) G = -\delta(M - M_0) \,,$$

where M_0 is the position of the source (we omit here an important question about behavior of desired solution as the observation point M tends to the infinity which actually provides uniqueness of the solution). Here also in a small vicinity of the source we may set a value for velocity by $C(M_0)$ and consider the problem for the homogeneous medium

$$\left(\Delta + \frac{\omega^2}{C^2(0)}\right) G = -\delta(x)\delta(z) \, . \tag{5.51}$$

(M_0 is put at the origin).

Solution of the latter problem (5.51) is known and reads

$$G = \frac{i}{4} H_0^{(1)}(kr) \quad , \quad k = \frac{\omega}{C(0)} \, ,$$

where $H_0^{(1)}$ is Hankel's function of zero order and of the first kind.

Suppose now that we are far from the source, precisely, that $kr \gg 1$. It implies that

$$kr = \frac{2\pi}{\lambda} r \gg 1$$

and we are far from the source in terms of numbers of wavelength!

Then we may replace $H_0^{(1)}(kr)$ by its asymptotics

$$H_0^{(1)}(kr) \simeq \sqrt{\frac{2}{\pi k r}} e^{i(kr - \frac{\pi}{4})}$$

and obtain

$$G \cong \frac{i e^{-i\frac{\pi}{4}}}{4} \sqrt{\frac{2}{\pi k}} \frac{e^{ikr}}{\sqrt{r}} = \frac{e^{i\frac{\pi}{4}}}{2\sqrt{2\pi}} \sqrt{\frac{C(0)}{\omega}} \frac{\exp\left[i\frac{\omega}{C(0)} r\right]}{\sqrt{r}} \, . \tag{5.52}$$

By comparing expressions (5.50) and (5.52) we obtain the following formula for $\psi_0(\alpha)$ in the case under consideration

$$\psi_0(\alpha) = \frac{e^{i\frac{\pi}{4}}}{2\sqrt{2\pi\omega}} \, . \tag{5.53}$$

Note that for the point source problem, ψ_0 does not depend on the ray parameter!

Consider now a 3D case.

In this case the Green's function G, i.e., solution of equation (5.51), is the following

$$G = \frac{1}{4\pi r} e^{ikr} \, , \quad k = \frac{\omega}{C(0)} \, , \quad r = \sqrt{x^2 + y^2 + z^2} \, . \tag{5.54}$$

On the other hand, for the central ray field in 3D and for a homogeneous medium we have

$$U \simeq \frac{\psi_0(\theta, \varphi)}{\sqrt{\frac{1}{C(0)} s^2 \sin\theta}} \exp\left(i \frac{\omega}{C(0)} s\right) = \frac{\psi_0(\theta, \varphi)}{\sqrt{\frac{1}{C(0)} \sin\theta}} \frac{e^{ikr}}{r} \, , \tag{5.55}$$

where θ and φ are the angles of the spherical coordinates.

By comparing (5.54) and (5.55) we obtain the following formula for $\psi_0(\theta, \varphi)$

$$\psi_0(\theta, \varphi) = \frac{1}{4\pi} \sqrt{\frac{\sin\theta}{C(0)}} \,,$$

and the initial amplitude ψ_0 does depend upon the ray parameter θ.

6

The ray method in a medium with smooth interfaces

When we study the wave propagation in a compound medium consisting of two or more different media with different characteristics, e.g. with different velocities, separated by smooth surfaces, we say that we are dealing with a media containing smooth interfaces. In this case, special type of boundary conditions on the wave field has to be imposed on the interfaces in order to describe the wave propagation in the whole medium. For the wave equation under consideration the classical boundary conditions on a smooth interface S are the following:

Dirichlet's condition

$$U\big|_S = 0 \tag{6.1}$$

Neumann's condition

$$\left.\frac{\partial U}{\partial n}\right|_{S=0} = 0 \quad , \tag{6.2}$$

where n is a coordinate along a normal vector to the surface S, and U means the total wave field.

The mixed boundary conditions describing a contact of two media read

$$U_1\big|_S = U_2\big|_S \quad \text{and} \quad \left.\frac{1}{\rho_1}\frac{\partial U_1}{\partial n}\right|_S = \left.\frac{1}{\rho_2}\frac{\partial U_2}{\partial n}\right|_S \ , \tag{6.3}$$

where U_1, U_2 are the total wave field values, and ρ_1, ρ_2 are physical properties, e.g. density values, in the first and in the second medium, respectively.

These conditions have a different physical sense but our main goal now is to study how to satisfy them within the frames of the ray theory.

Suppose, that the Dirichlet's condition is imposed on an interface S. Assume we have an incident wave given in the form

$$U^{(in)} = e^{i\omega\tau_{in}} A_0^{(in)} \,. \tag{6.4}$$

To satisfy (6.1) we suppose that the wave process near the boundary gives rise to a reflected wave $U^{(r)}$ which can be presented in the form of the ray series too

$$U^{(r)} = e^{i\omega\tau_r} A_0^{(r)} \,. \tag{6.5}$$

Now a new problem arises. How can the reflected eikonal, τ_r, and amplitude, $A_0^{(r)}$, be constructed when the incident eikonal τ_{in} and $A_0^{(in)}$ are known?

By inserting (6.4) and (6.5) in equation (6.1) we get

$$\left(e^{i\omega\tau_{in}} A_0^{(in)} + e^{i\omega\tau_r} A_0^{(r)}\right)\Big|_S = 0 \,. \tag{6.6}$$

In order to satisfy this equation we impose the requirement that separately

$$\tau_{in}\big|_S = \tau_r\big|_S \,, \tag{6.7}$$

and

$$(A_0^{(in)} + A_0^{(r)})\big|_S = 0 \,. \tag{6.8}$$

Similarly we get in the case of Neumann's condition

$$\left\{e^{i\omega\tau_{in}}\left(i\omega\frac{\partial\tau_{in}}{\partial n}A_0^{(in)} + \frac{\partial A_0^{(in)}}{\partial n}\right) + e^{i\omega\tau_r}\left(i\omega\frac{\partial\tau_r}{\partial n}A_0^{(r)} + \frac{\partial A_0^{(r)}}{\partial n}\right)\right\}\Bigg|_S = 0 \,,$$

and the corresponding requirement takes the form

$$\tau_{in}\big|_S = \tau_r\big|_S$$

and

$$\left(\frac{\partial\tau_{in}}{\partial n}A_0^{(in)} + \frac{\partial\tau_r}{\partial n}A_0^{(r)}\right)\Bigg|_S = 0 \,.$$

Observe that we omitted the term $\partial A_0/\partial n$ for both the incident and the reflected waves. Because they are not multiplied by the large parameter ω, they are supposed to be small.

In the case of conditions (6.3) we suppose that there are three waves: the incident and the reflected one in the first medium and a transmitted wave in the second medium. Hence, we get

$$\left(e^{i\omega\tau_{in}} A_0^{(in)} + e^{i\omega\tau_r} A_0^{(r)}\right)\Big|_S = e^{i\omega\tau_{tr}} A_0^{(tr)}\big|_S$$

and

$$\frac{1}{\rho_1}\left\{e^{i\omega\tau_{in}}\left(i\omega\frac{\partial\tau_{in}}{\partial n}A_0^{(in)} + \frac{\partial A_0^{(in)}}{\partial n}\right) + e^{i\omega\tau_r}\left(i\omega\frac{\partial\tau_r}{\partial n}A_0^{(r)} + \frac{\partial A_0^{(r)}}{\partial n}\right)\right\}\Bigg|_S =$$

$$= \frac{1}{\rho_2}e^{i\omega\tau_{tr}}\left(i\omega\frac{\partial\tau_{tr}}{\partial n}A_0^{(tr)} + \frac{\partial A_0^{(tr)}}{\partial n}\right)\Bigg|_S \,.$$

In the ray theory we substitute them by the following equations

$$\tau_{in}\big|_S = \tau_r\big|_S = \tau_{tr}\big|_S$$

and

$$\begin{aligned}\left(A_0^{(in)} + A_0^{(r)}\right)\Big|_S &= A_0^{(tr)}\big|_S\,,\\ \frac{1}{\rho_1}\left\{\frac{\partial\tau_{in}}{\partial n}A_0^{(in)} + \frac{\partial\tau_r}{\partial n}A_0^{(r)}\right\}\Bigg|_S &= \frac{1}{\rho_2}\frac{\partial\tau_{tr}}{\partial n}A_0^{(tr)}\Bigg|_S\,.\end{aligned}$$

It should be emphasized once again that, while deriving the boundary conditions within the zero-order term of the ray theory, we take into account only the main terms, i.e., the terms which contain the large parameter ω, and omit others. Hence, we satisfy the original boundary conditions only approximately in the ray theory.

And finally, as it follows from the theory of the eikonal and transport equations described above, in order to find the eikonals and amplitudes for the reflected and the transmitted waves, we have to construct proper families of rays.

6.1 Equations for the eikonals; Snell's law

Let us introduce in a vicinity of the incident point M three mutually orthogonal unit vectors $\vec{n}$, $\vec{l}_1$, $\vec{l}_2$ where $\vec{n}$ is a normal to the surface S and $\vec{l}_1$, $\vec{l}_2$ are placed on the tangent plane to S at the point M, see Fig. 6.1.

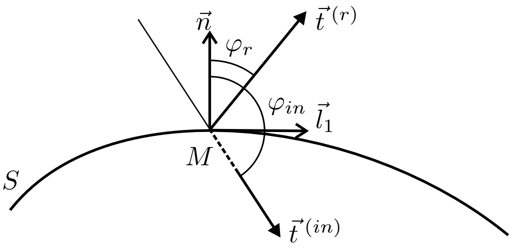

Figure 6.1: Reflection on an interface S. $\vec{t}^{\,(in)}$ and $\vec{t}^{\,(r)}$ are unit vectors tangent to the incident and the reflected rays, respectively.

By differentiating equation (6.7) for the eikonals in the direction of $\vec{l}_1$ and $\vec{l}_2$ we get

$$\frac{\partial\tau_{in}}{\partial l_j}\Bigg|_S = \frac{\partial\tau_r}{\partial l_j}\Bigg|_S \Rightarrow (\nabla\tau_{in}, \vec{l}_j) = (\nabla\tau_r, \vec{l}_j) \quad , j = 1, 2. \tag{6.9}$$

Note that we cannot differentiate equation (6.7) along the normal $\vec{n}$, because this equation is valid only on S!

It follows from (6.9) that the projections of $\nabla\tau_{in}$ and $\nabla\tau_r$ onto the tangent plane are equal and, hence, we get

$$\nabla\tau_{in} - \nabla\tau_r = \chi\vec{n}\,, \tag{6.10}$$

where χ is a scalar factor, $\chi \neq 0$. Taking into account that

$$\nabla\tau_{in} = \frac{\vec{t}^{\,(in)}}{C} \quad \text{and} \quad \nabla\tau_r = \frac{\vec{t}^{\,(r)}}{C}\,,$$

we get from (6.10) the first statement of the Snell's law. The three vectors $\vec{t}^{\,(in)}$, $\vec{t}^{\,(r)}$ and $\vec{n}$ are placed on one plane. This plane is called the reflection plane.

By multiplying (6.10) by $\vec{n}$ we obtain the following expression for χ

$$\frac{(\vec{t}^{\,(in)},\vec{n})}{C} - \frac{(\vec{t}^{\,(r)},\vec{n})}{C} = \chi \to \frac{\cos\varphi_{in}}{C} - \frac{\cos\varphi_r}{C} = \chi\,.$$

From the eikonal equation $(\nabla\tau)^2 = 1/C^2$ we derive

$$\begin{aligned}\frac{1}{C^2} - (\nabla\tau_{in},\vec{n})^2 &= (\nabla\tau_{in},\vec{l}_1)^2 + (\nabla\tau_{in},\vec{l}_2)^2 = (\nabla\tau_r,\vec{l}_1)^2 + (\nabla\tau_r,\vec{l}_2)^2 = \\ &= \frac{1}{C^2} - (\nabla\tau_r,\vec{n})^2\end{aligned}$$

and then

$$\frac{1}{C^2} - \frac{(\vec{t}^{\,(in)},\vec{n})^2}{C^2} = \frac{1}{C^2} - \frac{(\vec{t}^{\,(r)},\vec{n})^2}{C^2} \Rightarrow \frac{1-\cos^2\varphi_{in}}{C^2} = \frac{1-\cos^2\varphi_r}{C^2} \Rightarrow$$
$$\frac{\sin\varphi_{in}}{C} = \frac{\sin\varphi_r}{C}\,. \tag{6.11}$$

Equation (6.11) is the second statement of Snell's law.

Let us denote by C_1 and C_2 the velocity in the first and second media, respectively, then for the transmitted wave we obtain quite similarly

$$\frac{\sin\varphi_{in}}{C_1} = \frac{\sin\varphi_{tr}}{C_2}\,. \tag{6.12}$$

If, for instance, $C_2 > C_1$ we may get a situation when $\varphi_{tr} = \pi/2$. In this case $\sin\varphi_{in} = C_1/C_2$ and angle φ_{in} is called the critical angle. We shall see later that in this case the transmitted wave cannot be described by the ray series.

6.2 Equations for the amplitudes, reflection and transmission coefficients

For Dirichlet's boundary condition we get

$$A_0^{(r)}\big|_S = -A_0^{(in)}\big|_S \equiv RA_0^{(in)}\big|_S$$

where $R = -1$ and is called a reflection coefficient.

For Neumann's condition we obtain

$$A_0^{(r)}\big|_S = -\frac{(\nabla\tau_{in}, \vec{n})}{(\nabla\tau_r, \vec{n})} A_0^{(in)}\big|_S = A_0^{(in)}\big|_S$$

because

$$(\nabla\tau_{in}, \vec{n}) = \frac{1}{C}(\vec{t}^{\,(in)}, \vec{n}) = -\frac{1}{C}(\vec{t}^{\,(r)}, \vec{n}) \,.$$

A more complicated system of equations appears in the case of a contact between two media

$$(A_0^{(in)} + A_0^{(r)})\big|_S = A_0^{(tr)}\big|_S \,,$$

$$\frac{1}{\rho_1}\frac{(\vec{t}^{\,(in)}, \vec{n})}{C_1}(A_0^{(in)} - A_0^{(r)})\big|_S = \frac{1}{\rho_2}\frac{(\vec{t}^{\,(tr)}, \vec{n})}{C_2} A_0^{(tr)}\big|_S \,.$$

It can be written in the form

$$\begin{cases} A_0^{(tr)} - A_0^{(r)} = A_0^{(in)} \\ \dfrac{\cos\varphi_{tr}}{\rho_2 C_2} A_0^{(tr)} + \dfrac{\cos\varphi_{in}}{\rho_1 C_1} A_0^{(r)} = \dfrac{\cos\varphi_{in}}{\rho_1 C_1} A_0^{(in)} \end{cases} .$$

If the determinant of the system is not equal to 0, i.e.

$$\Delta = \frac{\cos\varphi_{in}}{\rho_1 C_1} + \frac{\cos\varphi_{tr}}{\rho_2 C_2} \neq 0$$

we obtain the unique solution of the system

$$\begin{aligned} A_0^{(tr)} &= 2A_0^{(in)} \frac{1}{\Delta} \frac{\cos\varphi_{in}}{\rho_1 C_1} = 2A_0^{(in)} \frac{1}{1 + \dfrac{\rho_1 C_1}{\rho_2 C_2}\dfrac{\cos\varphi_{tr}}{\cos\varphi_{in}}} \,, \\ A_0^{(r)} &= A_0^{(in)} \left(\frac{\cos\varphi_{in}}{\rho_1 C_1} - \frac{\cos\varphi_{tr}}{\rho_2 C_2} \right) \frac{1}{\Delta} = \\ &= A_0^{(in)} \frac{1 - \dfrac{\rho_1 C_1}{\rho_2 C_2}\dfrac{\cos\varphi_{tr}}{\cos\varphi_{in}}}{1 + \dfrac{\rho_1 C_1}{\rho_2 C_2}\dfrac{\cos\varphi_{tr}}{\cos\varphi_{in}}} \,. \end{aligned}$$

By introducing reflection R and transmission T coefficients through the formulas

$$A_0^{(r)} = R A_0^{(in)} \quad \text{and} \quad A_0^{(tr)} = T A_0^{(in)} \,,$$

we get from the latter results

$$R = \frac{1 - \dfrac{\rho_1 C_1}{\rho_2 C_2}\dfrac{\cos\varphi_{tr}}{\cos\varphi_{in}}}{1 + \dfrac{\rho_1 C_1}{\rho_2 C_2}\dfrac{\cos\varphi_{tr}}{\cos\varphi_{in}}} \,, \quad T = \frac{2}{1 + \dfrac{\rho_1 C_1}{\rho_2 C_2}\dfrac{\cos\varphi_{tr}}{\cos\varphi_{in}}} \,.$$

Note that $1 + R = T$.

Thus, to construct either the reflected or the refracted wave arising on the interface, we have to find either the reflected or the refracted ray for each incident ray and extend the eikonal continuously along it. This procedure is based on Snell's law - see equations (6.11), (6.12). To find the amplitude of the corresponding wave, we have to know either the reflection or the transmission coefficient which is caused by a particular boundary condition imposed on the interface.

6.3 Initial data for the geometrical spreading on an interface

To complete the computational algorithm for the geometrical spreading, we have to find the initial data for the solutions of the equations in variations along the reflected and the refracted central rays of a ray tube. There are several ways to approach the problem. We use a way based on the following conditions for the eikonals on an interface S:

$$\tau_{in}\big|_S = \tau_r\big|_S = \tau_{tr}\big|_S \, . \tag{6.13}$$

The main idea of the approach can be described as follows.

In a vicinity of the central ray of a ray tube, the eikonal can be presented in the form

$$\tau(s, q_1, q_2) = \tau_0(s) + \frac{1}{2}\sum_{i,j=1}^{2} \Gamma_{ij}(s) q_i q_j + \cdots \, . \tag{6.14}$$

If now we satisfy equations (6.13) for τ_{in}, τ_r and τ_{tr}, within accuracy up to the second order term in a vicinity of the incident point on the interface S, we shall find the relations among the matrices $\mathbf{\Gamma}$.

But we know that $\mathbf{\Gamma}$ can be presented as $\mathbf{P}\mathbf{Q}^{-1}$ and $|\det \mathbf{Q}|$ is the geometrical spreading. Then by transforming the $\mathbf{\Gamma}$ relations into $\mathbf{P}$ and $\mathbf{Q}$ relations, we shall find the desired initial data for the matrices $\mathbf{P}$ and $\mathbf{Q}$ for the reflected and the transmitted central rays at the point of incidence.

For the sake of simplicity, we consider this problem in detail for a 2D case. Final results in a 3D case are presented in the last section of this chapter.

Instead of equation (6.14) we have for a 2D case

$$\tau(s, q) = \tau_0(s) + \frac{1}{2}\Gamma q^2 + \cdots \tag{6.15}$$

and a similar expansion holds for τ_r and τ_{tr} but in local coordinates connected with the reflected and transmitted central rays.

Suppose that for $s = s_*$ the incident central ray $\vec{r}_0(s)$ intersects the interface S at the point M_*. Point M_* is called the point of incidence.

Let us introduce local coordinates ν, ζ at the point M_* where $\vec{\nu}$ is directed along the normal to S and $\vec{\zeta}$ is tangent to S at M_*, see Fig. 6.2.

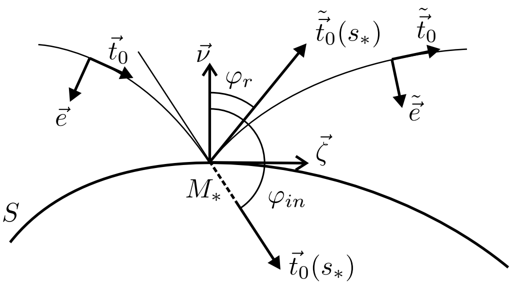

Figure 6.2: Local coordinates in a vicinity of a point of incidence M_* on the interface S.

We shall suppose further that the interface is given in the form

$$\nu = \frac{1}{2} D\zeta^2 + \cdots \tag{6.16}$$

because only second order terms will be involved in the calculations.

We shall have to calculate τ in a small vicinity of point M_*, meaning that $s - s_*$ is small along with other coordinates q, ν, ζ.

Hence, we can re-expand τ in power series on $s - s_*$ as well

$$\tau(s,q) = \tau_0(s_*) + \left.\frac{d\tau_0}{ds}\right|_{s_*} (s - s_*) + \left.\frac{d^2\tau_0}{ds^2}\right|_{s_*} \frac{(s-s_*)^2}{2} + \frac{1}{2}\Gamma(s_*)q^2 + \cdots . \tag{6.17}$$

The first step on our way consists of studying connecting formulas between s, q and ν, ζ in a vicinity of M_*.

To this end let us expand $\vec{r}_0(s)$

$$\begin{aligned} \vec{r}_0(s) &= \vec{r}_0(s_*) + \left.\frac{d\vec{r}_0}{ds}\right|_{s_*} (s - s_*) + \left.\frac{d^2\vec{r}_0}{ds^2}\right|_{s_*} \frac{(s-s_*)^2}{2} + \cdots = \\ &= \vec{r}_0(s_*) + \vec{t}_0(s_*)(s - s_*) - \kappa(s_*)\frac{(s-s_*)^2}{2}\vec{e}(s_*) + \cdots \end{aligned}$$

due to

$$\frac{d\vec{e}}{ds} = \kappa(s)\vec{t}_0 \quad \text{and} \quad \frac{d\vec{t}_0}{ds} = -\kappa(s)\vec{e}\,.$$

Thus, for the radius-vector $\vec{r}$ we obtain from the above

$$\begin{aligned} \vec{r} &= \vec{r}_0(s) + q\vec{e}(s) = \vec{r}_0(s_*) + \vec{t}_0(s_*)[(s - s_*) + q(s - s_*)\kappa(s_*) + \cdots] + \\ &+ \vec{e}(s_*)\left[q - \frac{(s-s_*)^2}{2}\kappa(s_*) + \cdots\right] . \end{aligned} \tag{6.18}$$

Evidently, at the same time we have

$$\vec{r} = \vec{r}_0(s_*) + \zeta\vec{\zeta} + \nu\vec{\nu} \tag{6.19}$$

where $\vec{r}_0(s_*)$ corresponds to the position of the origin of coordinates ζ, ν. Now from equations (6.18), (6.19) we obtain

$$\begin{cases} ls - s_* + (s - s_*)q\kappa(s_*) + \cdots = \zeta(\vec{\zeta}, \vec{t}_0(s_*)) + \nu(\vec{\nu}, \vec{t}_0(s_*)) \equiv \varepsilon_1 \\ q - (s - s_*)^2 \dfrac{\kappa(s_*)}{2} + \cdots = \zeta(\vec{\zeta}, \vec{e}(s_*)) + \nu(\vec{\nu}, \vec{e}(s_*)) \equiv \varepsilon_2 \end{cases} \tag{6.20}$$

where by ε_1 and ε_2 we denote the right-hand sides in the above equations.

Bearing our aim in mind, we need to know $s - s_*$ and q as functions of ζ and ν, so we have to solve the system (6.20) with respect to $s - s_*$ and q for small ε_1 and ε_2.

It is clear, that in the first approximation $s - s_* \simeq \varepsilon_1$ and $q \simeq \varepsilon_2$, so we may consider

$$\begin{aligned} s - s_* &= \varepsilon_1 + \delta_2 \, , \\ q &= \varepsilon_2 + \gamma_2 \, , \end{aligned} \tag{6.21}$$

where δ_2 and γ_2 contain the second order terms which have to be found. By inserting equation (6.21) into the system (6.20) we obtain after some algebra

$$\delta_2 = -\kappa(s_*)\varepsilon_1\varepsilon_2 \quad \text{and} \quad \gamma_2 = \frac{1}{2}\kappa(s_*)\varepsilon_1^2$$

and, eventually, we get

$$\begin{cases} s - s_* &= \varepsilon_1 - \kappa(s_*)\varepsilon_1\varepsilon_2 + \cdots \\ q &= \varepsilon_2 + \frac{1}{2}\kappa(s_*)\varepsilon_1^2 + \cdots \end{cases} . \tag{6.22}$$

Let us now come back to expansion (6.17) for τ. As we have q only in square, we may save for q the first order terms in equation (6.22) while for $s - s_*$ we have to save the second order terms as well. Then, according to equation (6.16), the coordinate ν is of a second order, so we can simplify the right-hand sides of equations (6.22) on the interface S.

$$\begin{cases} s - s_* = \zeta\,(\vec{\zeta}, \vec{t}_0(s_*)) + \frac{1}{2}D\,\zeta^2\,(\vec{\nu}, \vec{t}_0(s_*)) - \kappa(s_*)\,\zeta^2\,(\vec{\zeta}, \vec{t}_0(s_*))\,(\vec{\zeta}, \vec{e}(s_*)) + \cdots \\ q = \zeta\,\big(\vec{\zeta}, \vec{e}(s_*)\big) + \cdots \end{cases} \tag{6.23}$$

The system (6.23) is precisely the connecting formulas between the coordinates (s, q) and (ζ, ν) we are looking for. The connecting formulas have already been computed on the interface S by substituting $\nu = \frac{1}{2}D\zeta^2$.

Now we are able to accurately calculate the eikonal τ on the interface up to the second order terms

$$\begin{aligned}
\tau(s,q)\big|_S &= \tau_0(s_*) + \frac{d\tau_0}{ds}\bigg|_{s_*}\bigg[\zeta\,\Big(\vec{\zeta},\vec{t}_0(s_*)\Big) + \zeta^2\left(\frac{1}{2}D(\vec{\nu},\vec{t}_0(s_*)) -\right.\\
&\qquad \left. -\ \kappa(s_*)\,(\vec{\zeta},\vec{t}_0(s_*))\,(\vec{\zeta},\vec{e}(s_*))\right)\bigg] + \frac{d^2\tau_0}{ds^2}\bigg|_{s_*}\frac{1}{2}\zeta^2\,(\vec{\zeta},\vec{t}_0(s_*))^2 + \\
&\qquad + \frac{1}{2}\Gamma(s_*)\zeta^2\,(\vec{\zeta},\vec{e}(s_*))^2 + \cdots = \qquad (6.24)\\
&= \tau_0(s_*) + \zeta\,\frac{d\tau_0}{ds}\bigg|_{s_*}(\vec{\zeta},\vec{t}_0(s_*)) + \zeta^2\left\{\Gamma(s_*)\,\frac{(\vec{\zeta},\vec{e}(s_*))^2}{2} + \right.\\
&\qquad + \frac{d^2\tau_0}{ds^2}\bigg|_{s_*}\frac{(\vec{\zeta},\vec{t}_0(s_*))^2}{2} - \kappa(s_*)(\vec{\zeta},\vec{t}_0(s_*))(\vec{\zeta},\vec{e}(s_*))\,\frac{d\tau_0}{ds}\bigg|_{s_*} + \\
&\qquad \left. + \ D\frac{(\vec{\nu},\vec{t}_0(s_*))}{2}\,\frac{d\tau_0}{ds}\bigg|_{s_*}\right\} + \cdots .
\end{aligned}$$

Let us then carry out similar calculations for the refracted and the reflected central rays and the corresponding eikonals. But, in fact, we may write the final results down based on equations (6.23) and (6.24) because the computations are quite similar.

Let us supply by~all terms corresponding to the reflected (or refracted) central ray, so that, for instance, $\tilde{q}$, $\tilde{s}$ are the new ray centered coordinates. Based on formulas (6.23), the corresponding equations for coordinates $\tilde{q}$ and $\tilde{s} - s_*$ can be written immediately

$$\begin{aligned}
\tilde{q} &= \zeta\,(\vec{\zeta}(s_*),\tilde{\vec{e}}(s_*)) + \cdots , \qquad (6.25)\\
\tilde{s} - s_* &= \zeta\,(\vec{\zeta},\tilde{\vec{t}}_0(s_*)) + \zeta^2\left[\frac{D}{2}(\vec{\nu},\tilde{\vec{t}}_0(s_*)) - \tilde{\kappa}(s_*)(\vec{\zeta},\tilde{\vec{t}}_0(s_*))(\vec{\zeta},\tilde{\vec{e}}(s_*))\right] + \cdots .
\end{aligned}$$

Accordingly, either for the reflected or refracted eikonal $\tilde{\tau}(\tilde{s},\tilde{q})$ we get

$$\begin{aligned}
\tilde{\tau}(\tilde{s},\tilde{q})\big|_S &= \tilde{\tau}_0(s_*) + \zeta\frac{d\tilde{\tau}_0}{d\tilde{s}}\bigg|_{s_*}(\vec{\zeta},\tilde{\vec{t}}_0(s_*)) + \zeta^2\left\{\tilde{\Gamma}(s_*)\frac{1}{2}(\vec{\zeta},\tilde{\vec{e}}(s_*))^2 + \right.\\
&\qquad + \frac{d^2\tilde{\tau}_0}{d\tilde{s}^2}\bigg|_{s_*}\frac{1}{2}(\vec{\zeta},\tilde{\vec{t}}_0(s_*))^2 - \tilde{\kappa}(s_*)(\vec{\zeta},\tilde{\vec{t}}_0(s_*))(\vec{\zeta},\tilde{\vec{e}}(s_*))\,\frac{d\tilde{\tau}_0}{ds}\bigg|_{s_*} + \\
&\qquad \left. + \ D\frac{1}{2}(\vec{\nu},\tilde{\vec{t}}_0(s_*))\,\frac{d\tilde{\tau}_0}{ds}\bigg|_{s_*}\right\} + \cdots . \qquad (6.26)
\end{aligned}$$

Now we are able to accurately satisfy the basic equations $\tau_{in}\big|_s = \tau_r\big|_s = \tau_{tr}\big|_s$ up to the second order terms.

By comparing equations (6.24) and (6.26) we obtain for the main terms

$$\tilde{\tau}_0(s_*) = \tau_0(s_*) \quad . \qquad (6.27)$$

From the linear terms we get

$$\left.\frac{d\tilde{\tau}_0}{d\tilde{s}}\right|_{s_*}(\vec{\zeta},\tilde{\vec{t}}_0(s_*)) = \left.\frac{d\tau_0}{ds}\right|_{s_*}(\vec{\zeta},\vec{t}_0(s_*))\,. \tag{6.28}$$

The second order terms give rise to a relation between the matrices $\tilde{\Gamma}(s_*)$ and $\Gamma(s_*)$

$$\begin{aligned}
&\tilde{\Gamma}(s_*)\frac{1}{2}(\vec{\zeta},\tilde{\vec{e}}(s_*))^2 + \left.\frac{d^2\tilde{\tau}_0}{d\tilde{s}^2}\right|_{s_*}\frac{1}{2}(\vec{\zeta},\tilde{\vec{t}}_0(s_*))^2 - \tilde{\kappa}(s_*)(\vec{\zeta},\tilde{\vec{t}}_0(s_*))(\vec{\zeta},\tilde{\vec{e}}(s_*))\left.\frac{d\tilde{\tau}_0}{ds}\right|_{s_*} + \\
&D\frac{1}{2}(\vec{\nu},\tilde{\vec{t}}_0(s_*))\left.\frac{d\tilde{\tau}_0}{ds}\right|_{s_*} = \Gamma(s_*)\frac{1}{2}(\vec{\zeta},\vec{e}(s_*))^2 + \left.\frac{d^2\tau_0}{ds^2}\right|_{s_*}\frac{1}{2}(\vec{\zeta},\vec{t}_0(s_*))^2 - \\
&\left.\frac{d\tau_0}{ds}\right|_{s_*}\kappa(s_*)(\vec{\zeta},\vec{t}_0(s_*))(\vec{\zeta},\vec{e}(s_*)) + D\frac{1}{2}(\vec{\nu},\vec{t}_0(s_*))\left.\frac{d\tau_0}{ds}\right|_{s_*}\,.
\end{aligned} \tag{6.29}$$

Comments on equations (6.27) - (6.29).

Equation (6.27) means, quite naturally, that the eikonal has to be continuous on the central rays.

Let us check that equation (6.28) is already satisfied due to Snell's law. To this end consider, for example, the case of refraction.

Introduce $\gamma_{in} = \pi - \varphi_{in}$ and $\gamma_{tr} = \pi - \varphi_{tr}$, then, obviously,

$$(\vec{t}_0(s_*),\vec{\zeta}) = \cos(\pi - \frac{\pi}{2} - \gamma_{in}) = \sin\gamma_{in}\quad .$$

Accordingly,

$$(\tilde{\vec{t}}_0(s_*),\vec{\zeta}) = \cos(\frac{\pi}{2} - \gamma_{tr}) = \sin\gamma_{tr}$$

and, hence, equation (6.28) takes the form

$$\frac{1}{C_2(M_*)}\sin\gamma_{tr} = \frac{1}{C_1(M_*)}\sin\gamma_{in}$$

which is precisely Snell's law.

Consider further equation (6.29) in the case of a reflection. For that matter see also Fig.6.2.

According to the orientation of the local coordinates chosen at the point of incidence $s = s_*$ and the definition of the incident angle γ_{in}, we get the following formulas for the scalar products involved in equation (6.29)

$$\begin{aligned}
(\vec{\zeta},\tilde{\vec{t}}_0) &= (\vec{\zeta},\vec{t}_0) = \sin\gamma_{in}\\
(\vec{\nu},\tilde{\vec{t}}_0) &= -(\vec{\nu},\vec{t}_0) = \cos\gamma_{in}\\
(\vec{\zeta},\tilde{\vec{e}}) &= -(\vec{\zeta},\vec{e}) = \cos\gamma_{in}\,.
\end{aligned}$$

By inserting them into (6.29) we obtain

$$\tilde{\Gamma}(s_*)\frac{1}{2}\cos^2\gamma_{in} + \left.\frac{d^2\tilde{\tau}_0}{d\tilde{s}^2}\right|_{s_*}\frac{1}{2}\sin^2\gamma_{in} - \tilde{\kappa}(s_*)\sin\gamma_{in}\cos\gamma_{in}\left.\frac{d\tilde{\tau}_0}{ds}\right|_{s_*} + \frac{1}{2}D\cos\gamma_{in}\left.\frac{d\tilde{\tau}_0}{ds}\right|_{s_*} =$$
$$\Gamma(s_*)\frac{1}{2}\cos^2\gamma_{in} + \left.\frac{d^2\tau_0}{ds^2}\right|_{s_*}\frac{1}{2}\sin^2\gamma_{in} - \kappa(s_*)\sin\gamma_{in}(-\cos\gamma_{in})\left.\frac{d\tau_0}{ds}\right|_{s_*} + \frac{1}{2}D(-\cos\gamma_{in})\left.\frac{d\tau_0}{ds}\right|_{s_*}.$$

Then, after some algebra we get

$$\tilde{\Gamma}(s_*)\frac{1}{2}\cos^2\gamma_{in} = \Gamma(s_*)\frac{1}{2}\cos^2\gamma_{in} + \frac{1}{2}\sin^2\gamma_{in}\left(\left.\frac{d^2\tau_0}{ds^2}\right|_{s_*} - \left.\frac{d^2\tilde{\tau}_0}{d\tilde{s}^2}\right|_{s_*}\right) + \\ + \frac{1}{C(M_*)}(\kappa(s_*) + \tilde{\kappa}(s_*))\sin\gamma_{in}\cos\gamma_{in} - D\cos\gamma_{in}\frac{1}{C(M_*)}\,. \quad (6.30)$$

Now let us take into account that $\Gamma = P/Q$, where P and Q are to be solutions of the equations in variations, and let us derive the initial conditions for them on the interface, i.e. at the point $s = s_* = \tilde{s}$.

It is obvious, that all incident and reflected (refracted) rays remain to be continuous on an interface. In the ray centered coordinates they are described by the functions $q = q(s, \gamma)$ and $\tilde{q} = \tilde{q}(\tilde{s}, \gamma)$, respectively. It follows from equations (6.23) and (6.25) that in order to achieve continuity for the rays on S, in the first approximation, q and $\tilde{q}$ should satisfy the following condition

$$\left.\frac{\tilde{q}}{(\vec{\zeta}, \tilde{\vec{e}}(s_*))}\right|_S = \zeta = \left.\frac{q}{(\vec{\zeta}, \vec{e}(s_*))}\right|_S .$$

Now, taking into account that Q and $\tilde{Q}$ are derivatives of q and $\tilde{q}$ with respect to a ray parameter γ, we arrive at the following equation

$$\tilde{Q}(s_*) = \frac{(\vec{\zeta}, \tilde{\vec{e}}(s_*))}{(\vec{\zeta}, \vec{e}(s_*))}Q(s_*) \quad (6.31)$$

which provides a natural initial condition for $\tilde{Q}(\tilde{s})$ at the point $\tilde{s} = s_*$ on the interface S.

In the case of reflection, equation (6.31) yields

$$\tilde{Q}(s_*) = -Q(s_*) \quad (6.32)$$

and therefore we get from equation (6.30) the following relationship between $\tilde{P}$

and P

$$\tilde{P}(s_*) = -P(s_*) - \tan^2\gamma_{in}\left(\left.\frac{d^2\tau_0}{ds^2}\right|_{s_*} - \left.\frac{d^2\tilde{\tau}_0}{d\tilde{s}^2}\right|_{s_*}\right)Q(s_*) - \qquad (6.33)$$
$$-\frac{1}{C(M_*)}2\tan\gamma_{in}(\kappa(s_*)+\tilde{\kappa}(s_*))Q(s_*) + 2D\frac{1}{\cos\gamma_{in}}Q(s_*)\frac{1}{C(M_*)}\,.$$

Thus, equations (6.32) and (6.33) are the desirable initial data for $\tilde{Q}(s)$, $\tilde{P}(s)$ at the point of incidence on the interface. If we solve the equations in variations along the reflected central ray with the initial data given by equations (6.32), (6.33) we shall obtain the geometrical spreading on this ray.

It is convenient to present the final result in a matrix form

$$\begin{pmatrix} \tilde{Q}(s_*) \\ \tilde{P}(s_*) \end{pmatrix} = \mathbf{M}\begin{pmatrix} Q(s_*) \\ P(s_*) \end{pmatrix}$$

with the 2x2 matrix $\mathbf{M}$ having the following form

$$\mathbf{M} = \begin{pmatrix} -1 & 0 \\ M_{21} & -1 \end{pmatrix}$$

where

$$M_{21} = \frac{2D}{C(M_*)\cos\gamma_{in}} - \frac{\kappa(s_*)+\tilde{\kappa}(s_*)}{C(M_*)}2\tan\gamma_{in} - \tan^2\gamma_{in}\left.\left(\frac{d^2\tau_0}{ds^2} - \frac{d^2\tilde{\tau}_0}{d\tilde{s}^2}\right)\right|_{s_*}\,.$$

Note that in the case of a tangent incidence $\gamma_{in} = \pi/2$ and $\cos\gamma_{in} = 0$ and therefore M_{21} becomes singular. That means that the ray method cannot be used in this situation.

Remark. **J**-scalar product in 2D.

In case of a 2D problem, matrix **J** should be written as follows $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and column-vector X takes the form $X^{(j)} = \begin{pmatrix} Q_j \\ P_j \end{pmatrix}$, $j = 1, 2$.

Let us make sure that the reflection matrix $\mathbf{M}$ is a symplectic matrix, i.e $\mathbf{M}^T\mathbf{J}\mathbf{M} = \mathbf{J}$. Indeed, for the left-hand side of the latter equation we obtain consistently

$$\mathbf{M}^T\mathbf{J}\mathbf{M} = \begin{pmatrix} -1 & M_{21} \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ M_{21} & -1 \end{pmatrix} =$$
$$= \begin{pmatrix} -M_{21} & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ M_{21} & -1 \end{pmatrix} = \begin{pmatrix} M_{21} - M_{21} & 1 \\ -1 & 0 \end{pmatrix} =$$
$$= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \mathbf{J}\,.$$

It follows from that fact that the **J**-scalar product of two arbitrary solutions of the equations in variations is preserved for the reflections. It is not difficult to see that this result holds true for refraction as well.

6.4 Initial data for the geometrical spreading on an interface in 3D: the main results

A 3D medium equation of the interface in a vicinity of the point of incidence can be written in the following form

$$n = \frac{1}{2} \sum_{i,j=1}^{2} D_{ij} \zeta_i \zeta_j + \cdots , \tag{6.34}$$

where n is the distance to the interface along the normal and ζ_1, ζ_2 are some orthogonal coordinates on the tangent plane.

Instead of equation (6.17) we get, in this case, the following one

$$\begin{aligned} \tau(s, q_1, q_2) &= \tau_0(s_*) + \\ &+ \left.\frac{d\tau_0}{ds}\right|_{s_*} (s - s_*) + \left.\frac{d^2\tau_0}{ds^2}\right|_{s_*} \frac{(s-s_*)}{2} + \frac{1}{2} \sum_{i,j=1}^{2} \Gamma_{ij}(s_*) q_i q_j + \cdots . \end{aligned} \tag{6.35}$$

The next step consists of deriving connecting formulas between two coordinate systems: the local coordinates n, ζ_1, ζ_2 in a vicinity of the incident point on S and the ray centered coordinates s, q_1, q_2. These formulas come out from the vector equation

$$\vec{r}_0(s) + q_1 \vec{e}_1(s) + q_2 \vec{e}_2(s) = \vec{r}_0(s_*) + n\vec{n} + \zeta_1 \vec{\zeta}_1 + \zeta_2 \vec{\zeta}_2, \tag{6.36}$$

after accurately expanding the left-hand side into a power series with respect to $s - s_*$ up to the second order term.

After that we have to calculate τ on S for the incident, reflected and transmitted rays. By comparing these formulas for the eikonals, we obtain relations between the matrices $\mathbf{\Gamma}$, $\tilde{\mathbf{\Gamma}}$ at the incident point M_*. To get initial data for $\tilde{\mathbf{Q}}(s_*)$ and $\tilde{\mathbf{P}}(s_*)$ we have to add the condition in which the rays from the ray tube are to be continuous on the interface.

Final results can be presented in a matrix form.

Let us denote by $\tilde{X}^{(j)}, j = 1, 2$, solutions of the equations in variations on the reflected (transmitted) central ray, then at the point $s = s_*$ we get

$$\tilde{X}^{(j)}(s_*) = \mathbf{M} X^{(j)}(s_*) \tag{6.37}$$

where under $X^{(j)}(s)$ we understand the solutions of the equations in variations on the incident central ray, and $\mathbf{M}$ is a matrix 4x4. It contains derivatives of the velocities and coefficients D_{ij} from equation (6.34). It can be verified that $\mathbf{M}$ is a symplectic matrix and therefore the $\mathbf{J}$-scalar product is preserved for the reflected and refracted rays, i.e.

$$\left(\mathbf{J}\tilde{X}^{(1)}, \tilde{X}^{(2)}\right) = \left(\mathbf{J}X^{(1)}, X^{(2)}\right) .$$

Example. Reflection on a curved interface in a homogeneous medium; a 2D case.

Assume the interface is given by equation (6.16) in local coordinates in a vicinity of the point of incidence

$$\nu = \frac{1}{2}D\zeta^2 + \cdots .$$

The initial conditions on the interface take the form

$$\begin{cases} \tilde{Q}(s_*) = -Q(s_*) \\ \tilde{P}(s_*) = -P(s_*) + \frac{2D}{C\cos\gamma_{in}}Q(s_*) \end{cases}$$

due to $d^2\tau_0/ds^2 = d/ds(1/C) = 0$ and $\kappa \equiv 0$ in a homogeneous medium.

The equations in variations are the following

$$\begin{cases} \dfrac{d}{ds}Q = CP \\ \dfrac{d}{ds}P = 0 \end{cases} \quad \text{and} \quad \begin{cases} \dfrac{d}{d\tilde{s}}\tilde{Q} = C\tilde{P} \\ \dfrac{d}{d\tilde{s}}\tilde{P} = 0 \end{cases} .$$

In the case of a point source we had before $P(0) = 1/C$, $Q(0) = 0$ and therefore $Q(s) = s$, $P(s) = 1/C$. The general solution for $\tilde{Q}$ and $\tilde{P}$ reads $\tilde{Q}(\tilde{s}) = C\tilde{P}(\tilde{s} - s_*) + \tilde{Q}(s_*)$; $\tilde{P}(\tilde{s}) = \tilde{P}(s_*) = \text{const}$.

Hence, on the interface we obtain

$$\begin{aligned} \tilde{Q}(s_*) &= -Q(s_*) = -s_* \\ \tilde{P}(s_*) &= -\frac{1}{C} + \frac{2D}{C\cos\gamma_{in}}s_* . \end{aligned}$$

Finally,

$$\tilde{Q}(\tilde{s}) = \left(-1 + 2D\frac{s_*}{\cos\gamma_{in}}\right)(\tilde{s} - s_*) - s_*$$

and for the geometrical spreading on the reflected central ray we obtain the following formula

$$\tilde{J} = |\tilde{Q}(\tilde{s})| = \left| -(\tilde{s} - s_*) - s_* + 2D\frac{s_*(\tilde{s} - s_*)}{C\cos\gamma_{in}} \right| .$$

7

The ray method in elastodynamics

7.1 Plane waves in a homogeneous isotropic medium

Plane waves can be regarded as the simplest solution of the equations of motion which, at the same time, illuminate the main peculiarities of those equations in elastic homogeneous and isotropic media.

Let us start with the equations of motion in the form

$$(\lambda+\mu)\frac{\partial}{\partial x_j}\frac{\partial U_n}{\partial x_n}+\mu\frac{\partial^2 U_j}{\partial x_n \partial x_n}=\rho\frac{\partial^2 U_j}{\partial t^2}\ , \quad j=1,2,3\ , \tag{7.1}$$

and seek a solution in the following form

$$U_n=A_n e^{i\varphi}\ , \quad \varphi=-\omega t+k_m x_m\ , \tag{7.2}$$

where the vector-amplitude $\vec{A}$ is constant, and the phase function φ is a linear function of all arguments. This implies that ω, k_1, k_2, k_3 are also constant. It follows from formula (7.2) that

$$\frac{\partial U_j}{\partial x_n}=A_j e^{i\varphi} i\frac{\partial \varphi}{\partial x_n}\ , \quad \frac{\partial^2 U_j}{\partial x_n \partial x_n}=A_j e^{i\varphi}(i)^2\frac{\partial \varphi}{\partial x_n}\frac{\partial \varphi}{\partial x_n}\ ,$$

$$\frac{\partial}{\partial x_j}\frac{\partial U_n}{\partial x_n}=A_n e^{i\varphi}(i)^2\frac{\partial \varphi}{\partial x_n}\frac{\partial \varphi}{\partial x_j}\ .$$

By inserting the latter results into equations (7.1) we obtain

$$(\lambda+\mu)\frac{\partial \varphi}{\partial x_j}\frac{\partial \varphi}{\partial x_n}A_n+\mu\frac{\partial \varphi}{\partial x_n}\frac{\partial \varphi}{\partial x_n}A_j=\rho\frac{\partial \varphi}{\partial t}\frac{\partial \varphi}{\partial t}A_j\ , \quad j=1,2,3\ ,$$

and then

$$(\lambda+\mu)k_j k_n A_n + (\mu k_n k_n - \rho\omega^2)A_j = 0\,, \quad j = 1,2,3\,. \tag{7.3}$$

Let us introduce an additional notation

$$\Lambda = \mu k_n k_n - \rho\omega^2 = \mu|\vec{k}|^2 - \rho\omega^2\,.$$

Then a linear system of equations (7.3) for vector-amplitude $\vec{A}$ can be written in the matrix form

$$\mathbf{N}\vec{A} = 0 \tag{7.4}$$

where

$$\mathbf{N} = \begin{pmatrix} (\lambda+\mu)k_1k_1+\Lambda & (\lambda+\mu)k_1k_2 & (\lambda+\mu)k_1k_3 \\ (\lambda+\mu)k_2k_1 & (\lambda+\mu)k_2k_2+\Lambda & (\lambda+\mu)k_2k_3 \\ (\lambda+\mu)k_3k_1 & (\lambda+\mu)k_3k_2 & (\lambda+\mu)k_3k_3+\Lambda \end{pmatrix}.$$

It is well known that a linear system of homogeneous equations (7.4) has a nonzero solution if and only if $\det\mathbf{N} = 0$. We must then calculate $\det\mathbf{N}$.

Evidently, we can decompose $\det\mathbf{N}$ as 2^3 determinants, but only the following four of them do not vanish

$$\det\mathbf{N} = (\lambda+\mu)\begin{vmatrix} k_1k_1 & 0 & 0 \\ k_2k_1 & \Lambda & 0 \\ k_3k_1 & 0 & \Lambda \end{vmatrix} + (\lambda+\mu)\begin{vmatrix} \Lambda & k_1k_2 & 0 \\ 0 & k_2k_2 & 0 \\ 0 & k_3k_2 & \Lambda \end{vmatrix} +$$

$$+ (\lambda+\mu)\begin{vmatrix} \Lambda & 0 & k_1k_3 \\ 0 & \Lambda & k_2k_3 \\ 0 & 0 & k_3k_3 \end{vmatrix} + \begin{vmatrix} \Lambda & 0 & 0 \\ 0 & \Lambda & 0 \\ 0 & 0 & \Lambda \end{vmatrix} =$$

$$= (\lambda+\mu)k_1^2\Lambda^2 + (\lambda+\mu)k_2^2\Lambda^2 + (\lambda+\mu)k_2^3\Lambda^2 + \Lambda^3 = \Lambda^3 + \Lambda^2(\lambda+\mu)|\vec{k}|^2\,.$$

Hence, the characteristic equation $\det\mathbf{N} = 0$ takes the form

$$\det\mathbf{N} = \Lambda^2(\Lambda + (\lambda+\mu)|\vec{k}|^2) = 0$$

and gives rise to the following two possibilities.

1. A compressional wave:

$$\Lambda + (\lambda+\mu)|\vec{k}|^2 = 0 \Rightarrow (\lambda+2\mu)|\vec{k}|^2 - \rho\omega^2 = 0 \Rightarrow$$

$$\Rightarrow |\vec{k}|^2 = \frac{\omega^2}{\alpha^2}\,, \quad \alpha^2 = \frac{\lambda+2\mu}{\rho}\,, \quad \alpha = \sqrt{\frac{\lambda+2\mu}{\rho}}\,,$$

where α is the velocity of the compressional waves.

2. Shear waves:

$$\Lambda^2 = 0 \Rightarrow |\vec{k}|^2 = \frac{\omega^2}{\beta^2}\,, \quad \beta^2 = \frac{\mu}{\rho}\,, \quad \beta = \sqrt{\frac{\mu}{\rho}}\,,$$

where β is the velocity of the shear waves.

Next, we have to construct eigenvectors $\vec{A}$ for both cases 1) and 2).

Evidently, in case (1) we have only one eigenvector apart from the normalizing factor. Suppose further that $\vec{A}$ is constructed. By multiplying equation (7.4) by $\vec{A}$ we obtain:

$$\begin{aligned}(\mathbf{N}\vec{A},\vec{A}) &= \sum_{n,m} N_{nm}A_nA_m = \sum_{n,m}((\lambda+\mu)k_nk_m + \Lambda\delta_{nm})A_nA_m \\ &= \sum_{n,m}\big[(\lambda+\mu)k_nk_m - (\lambda+\mu)|\vec{k}|^2\delta_{nm}\big]A_nA_m \\ &= (\lambda+\mu)\big[(\vec{k},\vec{A})(\vec{k},\vec{A}) - |\vec{k}|^2(\vec{A},\vec{A})\big] = 0\,.\end{aligned}$$

Due to $\lambda + \mu \neq 0$ we get

$$(\vec{k},\vec{A})(\vec{k},\vec{A}) = |\vec{k}|^2|\vec{A}|^2\,,$$

which implies that the wave vector $\vec{k}$ and the amplitude $\vec{A}$ are collinear, i.e. $\vec{k} \uparrow\uparrow \vec{A}$ or $\vec{k} \downarrow\uparrow \vec{A}$.

Definition: We say, in this case, that the polarization of the compressional plane wave coincides with its direction of propagation fixed by the wave vector $\vec{k}$.

In case (2) we can construct two mutually orthogonal eigenvectors $\vec{A}$ due to matrix **N** being symmetrical. This fact is well known in linear algebra.

Suppose now that $\vec{A}$ satisfies equation (7.4), then by multiplying it by $\vec{A}$ and taking into account that $\Lambda = 0$, we obtain

$$(N\vec{A},\vec{A}) = \sum_{n,m} N_{nm}A_nA_m = (\lambda+\mu)\sum_{n,m} k_nk_mA_nA_m = (\lambda+\mu)(\vec{k},\vec{A})(\vec{k},\vec{A}) = 0$$

and therefore

$$(\vec{k},\vec{A}) = 0$$

which means that the polarization of the shear plane waves is orthogonal to the direction of propagation.

Hence, in this case we get two shear plane waves having orthogonal polarization.

Definition: The slowness vector $\vec{p}$ is defined by the formula $\vec{k} = \omega\vec{p}$.

Due to $|\vec{k}|^2 = \omega^2/c^2$ we get $|\vec{p}|^2 = 1/c^2$.

Let us now formulate the main results of the section.

In a homogeneous isotropic medium three plane waves may propagate in any direction fixed by the wave-vector $\vec{k}$. The fastest one is a compressional wave, or P-wave, its polarization being colinear with the direction of propagation. Two shear plane waves, or S-waves, have the same velocity of propagation, but mutually orthogonal polarization. Their polarization, in both cases, is orthogonal to the direction of propagation.

7.2 Reflection/transmission of plane waves on a plane interface

Similar to the plane wave propagation in an unbounded medium, the plane wave reflection/transmission on a plane interface is an important phenomenon not only for tutorial reasons, but also for important applications in geophysics. It should also be emphasized that the mathematical technique used for studying this phenomenon, in the case of plane waves, underlies, in fact, investigations of corresponding problems both in the ray theory and in the Gaussian Beam method.

If a medium under consideration contains interfaces some boundary conditions should be imposed on them.

The most typical and widespread boundary conditions in the theory of elasticity are the following:

1. an interface between two media,
2. a traction free surface.

In the first case, boundary conditions result usually from the wedded contact of both parts of the medium separated by an interface. This contact prevents diffusion of the material across the interface and the sliding of two solid media along the separating interface. In the second case, we may consider a contact between a solid medium and vacuum.

For the sake of simplicity, consider a traction free surface. We assume the interface to be a plane. The corresponding boundary conditions can be precisely satisfied by means of plane waves.

We consider the special Cartesian coordinates x_1, x_2, x_3 with the origin located on interface S and x_1 directed along a normal to S.

We shall denote the normal to S by $\vec{n}$ following our former notations. The incident plane wave $\vec{U}^{(in)}$ is assumed to be given in the x_1, x_2, x_3 coordinates.

The boundary conditions on a traction free interface S can be formulated as follows. Traction $\vec{T}(\vec{n})$ acting on the interface S with normal $\vec{n}$ is equal to zero, i.e. $T_i = \tau_{ji} n_j = 0$ for $i = 1, 2, 3$ where τ_{ji} are elements of the stress tensor. In our case $\vec{n} = (1, 0, 0)$ and therefore $T_i = \tau_{1i} = 0$. This means that

$$\tau_{11}|_S = 0\,, \quad \tau_{12}|_S = 0\,, \quad \tau_{13}|_S = 0\,. \tag{7.5}$$

In order to satisfy equations (7.5) we suppose that two reflected waves appear on S, namely, reflected S and P waves.

Let us introduce the following notations:

$$\begin{aligned} \vec{U}^{(in)} &= \vec{A}^{(in)} e^{i\varphi_{in}} \quad \text{for the incident wave,} \\ \vec{U}^{(p)} &= \vec{P}^{(r)} e^{i\varphi_{rp}} \quad \text{for the reflected } P\text{-wave,} \\ \vec{U}^{(s)} &= \vec{S}^{(r)} e^{i\varphi_{rs}} \quad \text{for the reflected } S\text{-wave,} \end{aligned}$$

where, of course, the phase functions φ are linear functions of their arguments, for instance,

$$\varphi_{in} = -\omega_{in} t + k_j^{in} x_j \quad \text{and so on.}$$

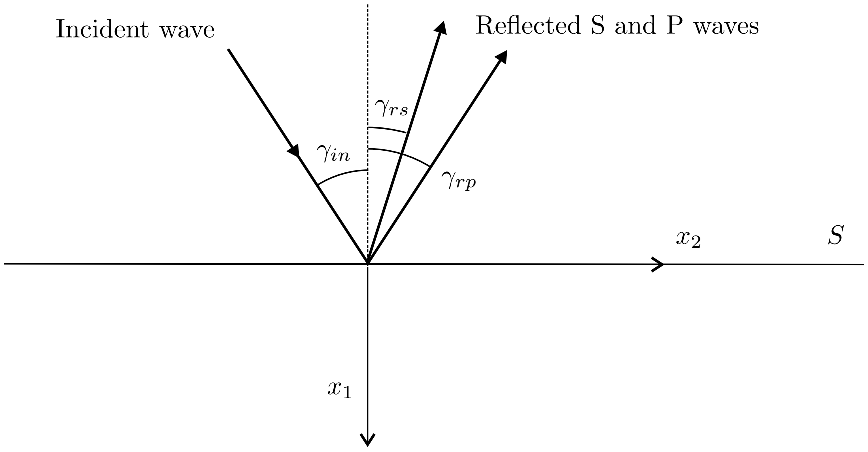

Figure 7.1: Reflection of an incident wave on a traction free plane interface.

The total displacement vector $\vec{U}$ takes the form

$$\vec{U} = \vec{U}^{(in)} + \vec{U}^{(p)} + \vec{U}^{(s)} . \tag{7.6}$$

For the stress tensor τ_{kj} we have the following formulas

$$\tau_{11} = \lambda \text{ div } \vec{U} + 2\mu \frac{\partial U_1}{\partial x_1} , \quad \tau_{1j} = \mu \left(\frac{\partial U_1}{\partial x_j} + \frac{\partial U_j}{\partial x_1} \right) , \quad j = 2, 3 . \tag{7.7}$$

Now we have to insert equations (7.6) and (7.7) into equation (7.5). Note that the equation of the interface S has the form

$$x_1 = 0 . \tag{7.8}$$

To this end let us look at any derivative of the displacement vector, for example,

$$\frac{\partial}{\partial x_m} U_j^{(in)} = A_j^{(in)} e^{i\varphi_{in}} i \frac{\partial \varphi_{in}}{\partial x_m} = i e^{i\varphi_{in}} A_j^{(in)} k_m^{in} .$$

Thus, after inserting such expressions into the boundary conditions we get $e^{i\varphi}$ for each wave and in order to satisfy equation (7.5), identically with respect to time and coordinates x_2, x_3 on the interface, we have to impose

$$\varphi_{in}\big|_{x_1=0} = \varphi_{rp}\big|_{x_1=0} = \varphi_{rs}\big|_{x_1=0} . \tag{7.9}$$

It follows immediately from equation (7.9) that

1. $\omega_{in} = \omega_{rp} = \omega_{rs} \equiv \omega$, i.e. circular frequency ω is preserved for the reflected waves.

2. The projection of the wave vector $\vec{k}$ onto the x_2, x_3- plane is the same for each wave, i.e.

$$\vec{k}^{in} - (\vec{k}^{in}, \vec{n})\vec{n} = \vec{k}^{rp} - (\vec{k}^{rp}, \vec{n})\vec{n} = \vec{k}^{rs} - (\vec{k}^{rs}, \vec{n})\vec{n} \tag{7.10}$$

giving rise to Snell's law.

Indeed, $\vec{k}^{in}, \vec{k}^{rp}, \vec{k}^{rs}$ and $\vec{n}$ belong to the same plane. Then, taking into account that $|\vec{k}| = \omega/C$, where C_{in} is the velocity of the incident wave, and

$$|\vec{k}|^2 - (\vec{k}, \vec{n})^2 = (\vec{k}, \vec{i}_2)^2 + (\vec{k}, \vec{i}_3)^2 \quad ,$$

where $\vec{i}_2, \vec{i}_3$ are unit vectors directed along x_2, x_3, respectively, we obtain from equation (7.10)

$$\frac{\omega^2}{C_{in}^2} \sin^2\langle\vec{k}^{in}, \vec{n}\rangle = \frac{\omega^2}{\alpha^2} \sin^2\langle\vec{k}^{rp}, \vec{n}\rangle = \frac{\omega^2}{\beta^2} \sin^2\langle\vec{k}^{rs}, \vec{n}\rangle \, . \tag{7.11}$$

In equations (7.11) we denote by $\langle\vec{k}, \vec{n}\rangle$ the angle between the wave vector $\vec{k}$ and the normal $\vec{n}$ to the interface. By introducing the incident angle γ_{in} and the reflected angles γ_{rp}, γ_{rs} we can rewrite equations (7.11) as follows

$$\frac{\sin\gamma_{in}}{C_{in}} = \frac{\sin\gamma_{rp}}{\alpha} = \frac{\sin\gamma_{rs}}{\beta} \, . \tag{7.12}$$

Note that equations (7.12) hold true both for the incident P-wave and the S-wave. To this end we have to replace velocity C_{in} by α or β, respectively.

7.3 Reflection coefficients

It is clear now that all exponents $e^{i\varphi}$ can be canceled out in equations (7.5) and therefore the equations lead to a linear system of algebraic equations for unknown vector-amplitudes of the reflected waves.

Consider first

$$\tau_{11} = \lambda \text{ div } \vec{U} + 2\mu \frac{\partial U_1}{\partial x_1} \, .$$

After some mathematics we obtain

$$\tau_{11}\big|_S = 0 \to \lambda\big(P_n^{(r)} k_n^{(rp)} + S_n^{(r)} k_n^{(rs)}\big) + 2\mu\big(P_1^{(r)} k_1^{(rp)} + S_1^{(r)} k_1^{(rs)}\big) = \\ = -\lambda A_n^{(in)} k_n^{(in)} - 2\mu A_1^{(in)} k_1^{(in)} \, .$$

Then, from $\tau_{1j}\big|_S = 0$ we get

$$P_1^{(r)} k_j^{(rp)} + P_j^{(r)} k_1^{(rp)} + S_1^{(r)} k_j^{(rs)} + S_j^{(r)} k_1^{(rs)} = -\big(A_1^{(in)} k_j^{in)} + A_j^{(in)} k_1^{(in)}\big) \, , \; j = 2, 3.$$

Using the scalar product, the first equation can be written in the form

$$\lambda\big\{\big(\vec{P}^{(r)},\vec{k}^{(rp)}\big)+\big(\vec{S}^{(r)},\vec{k}^{(rs)}\big)\big\}+2\mu\big(P_1^{(r)}k_1^{rp)}+S_1^{(r)}k_1^{(rs)}\big)= \\ =-\lambda\big(\vec{A}^{(in)},\vec{k}^{(in)}\big)-2\mu A_1^{(in)}k_1^{(in)}\,,$$

where, obviously, $\big(\vec{S}^{(r)},\vec{k}^{(rs)}\big)=0$ for the shear wave.

Let us consider further a particular case when the incident wave is a P wave and the incident plane coincides with the x_1, x_2 - plane, see Fig. 7.2.

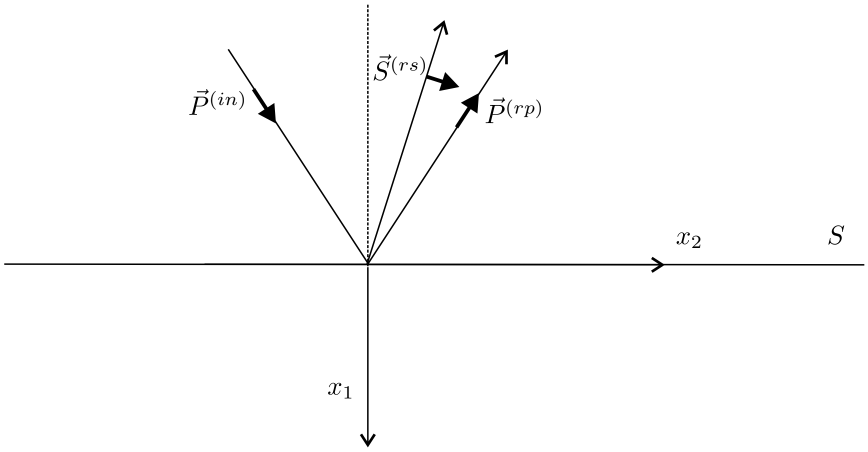

Figure 7.2: Reflection of an incident P-wave on the traction free plane interface S and the position of the polarization vectors.

Evidently, in this case we have

$$k_3^{(rp)}=0,\quad k_3^{(in)}=0,\quad k_3^{(rs)}=0,\quad P_3^{(r)}=0,\quad A_3^{(in)}=0\,.$$

Hence, from $\tau_{13}\big|_S=0$ we obtain $S_3^{(r)}k_1^{(rs)}=0$ and therefore $S_3^{(r)}=0$ which means precisely that the vector $\vec{S}^{(r)}$ belongs to the plane of incidence too!

Now let us consider the system of linear equations in more details in order to find the reflection coefficients.

It is convenient to re-denote $\vec{A}^{(in)}$ by $\vec{P}^{(in)}$. From $\tau_{11}|_S=0$ we get

$$\lambda\big(\vec{P}^{(r)},\vec{k}^{(rp)}\big)+2\mu\big(P_1^{(r)}k_1^{rp)}+S_1^{(r)}k_1^{(rs)}\big)= \\ =-\lambda\big(\vec{P}^{(in)},\vec{k}^{(in)}\big)-2\mu P_1^{(in)}k_1^{(in)}\,. \tag{7.13}$$

Then, equation $\tau_{13}\big|_S=0$ is already satisfied.

Equation $\tau_{12}\big|_S=0$ takes the form

$$P_1^{(r)}k_2^{(rp)}+P_2^{(r)}k_1^{(rp)}+S_1^{(r)}k_2^{(rs)}+S_2^{(r)}k_1^{(rs)}= \\ =-\big(P_1^{(in)}k_2^{in)}+P_2^{(in)}k_1^{(in)}\big) \tag{7.14}$$

Now let us introduce the incident angle γ_{in} and the reflected angles γ_{rp} and γ_{rs}.

For P - waves we get:

$$\begin{aligned} \gamma_{in} &= \gamma_{rp} \\ \left|\vec{k}^{(in)}\right| &= \left|\vec{k}^{(rp)}\right| = \frac{\omega}{\alpha} \\ \left(\vec{P}^{(in)}, \vec{k}^{(in)}\right) &= \left|\vec{P}^{(in)}\right|\left|\vec{k}^{(in)}\right| \\ P_1^{(in)} &= \left|\vec{P}^{(in)}\right| \cos\gamma_{in}; P_2^{(in)} = \left|\vec{P}^{(in)}\right| \sin\gamma_{in} \\ k_1^{(in)} &= \left|\vec{k}^{(in)}\right| \cos\gamma_{in}; k_2^{(in)} = \left|\vec{k}^{(in)}\right| \sin\gamma_{in} . \end{aligned} \tag{7.15}$$

For the reflected P-wave we obtain

$$\begin{aligned} \left(\vec{P}^{(r)}, \vec{k}^{(rp)}\right) &= \left|\vec{P}^{(r)}\right|\left|\vec{k}^{(rp)}\right| \\ P_1^{(r)} &= -\left|\vec{P}^{(r)}\right| \cos\gamma_{rp} = -\left|\vec{P}^{(r)}\right| \cos\gamma_{in} \\ k_1^{(rp)} &= -\left|\vec{k}^{(rp)}\right| \cos\gamma_{rp} = -\left|\vec{k}^{(rp)}\right| \cos\gamma_{in} \\ P_2^{(r)} &= \left|\vec{P}^{(r)}\right| \sin\gamma_{rp} = \left|\vec{P}^{(r)}\right| \sin\gamma_{in} \\ k_2^{(rp)} &= \left|\vec{k}^{(rp)}\right| \sin\gamma_{rp} = \left|\vec{k}^{(rp)}\right| \sin\gamma_{in} . \end{aligned} \tag{7.16}$$

For the reflected S - wave we get the following results:

$$\begin{aligned} S_1^{(r)} &= \left|\vec{S}^{(r)}\right| \cos\left(\frac{\pi}{2} - \gamma_{rs}\right) = \left|\vec{S}^{(r)}\right| \sin\gamma_{rs} \\ k_1^{(rs)} &= -\left|\vec{k}^{(rs)}\right| \cos\gamma_{rs} \\ S_2^{(r)} &= \left|\vec{S}^{(r)}\right| \cos\gamma_{rs} \\ k_2^{(rs)} &= \left|\vec{k}^{(rs)}\right| \sin\gamma_{rs} . \end{aligned} \tag{7.17}$$

By inserting equations (7.15) - (7.17) into equation (7.13) we obtain

$$\begin{aligned} \left|\vec{P}^{(r)}\right|\left|\vec{k}^{(rp)}\right|(\lambda + 2\mu\cos^2\gamma_{in}) - \left|\vec{S}^{(r)}\right|\left|\vec{k}^{(rs)}\right|\mu\sin 2\gamma_{rs} &= \\ = -\left|\vec{P}^{(in)}\right|\left|\vec{k}^{(in)}\right|(\lambda + 2\mu\cos^2\gamma_{in}) \end{aligned} \tag{7.18}$$

From equation (7.14) we get

$$\left|\vec{P}^{(r)}\right|\left|\vec{k}^{(rp)}\right| \sin 2\gamma_{rp} + \left|\vec{S}^{(r)}\right|\left|\vec{k}^{(rs)}\right| \cos 2\gamma_{rs} = \left|\vec{P}^{(in)}\right|\left|\vec{k}^{(in)}\right| \sin 2\gamma_{in} . \tag{7.19}$$

Thus we arrive at the linear system of algebraic equations with determinant Δ being equal to

$$\begin{aligned} \Delta &= \left|\vec{k}^{(rp)}\right|\left|\vec{k}^{(rs)}\right| \left[(\lambda + 2\mu\cos^2\gamma_{in})\cos 2\gamma_{rs} + \mu\sin 2\gamma_{rs}\sin 2\gamma_{rp}\right] = \\ &= \frac{\omega^2}{\alpha\beta} \left[(\lambda + 2\mu\cos^2\gamma_{in})\cos 2\gamma_{rs} + \mu\sin 2\gamma_{rs}\sin 2\gamma_{rp}\right] . \end{aligned}$$

By solving the linear system of equations (7.18) and (7.19) with respect to $\vec{P}^{(r)}$ and $\vec{S}^{(r)}$ we obtain the following results

$$\left|\vec{P}^{(r)}\right| = R_{pp}\left|\vec{P}^{(in)}\right| ; \quad \left|\vec{S}^{(r)}\right| = R_{sp}\left|\vec{P}^{(in)}\right| , \tag{7.20}$$

where R_{pp} and R_{sp} are called the reflection coefficients and

$$\begin{aligned} R_{pp} &= \frac{\omega^2}{\Delta\alpha\beta}\left[\mu \sin 2\gamma_{in} \sin 2\gamma_{rs} - (\lambda + 2\mu \sin^2 \gamma_{in}) \cos 2\gamma_{rs}\right] , \\ R_{sp} &= \frac{\omega^2}{\Delta\alpha^2} 2(\lambda + \mu) \sin 2\gamma_{in} . \end{aligned} \tag{7.21}$$

Thus, we observe that the traction free plane interface being impinged by a plane P wave gives rise to two reflected plane P and S waves. They jointly and exactly satisfy the boundary conditions on the interface. Clearly, the reflected P and S-waves are uniquely constructed.

To summarize the final results let us introduce the following definitions:

a) The shear wave polarized in the plane of incidence is called SV - wave (its polarization vector is on the vertical plane, i.e. V - plane).

b) The shear wave polarized perpendicularly to the plane of incidence is called SH - wave (its polarization is on the horizontal plane, i.e. H - plane).

Free surface reflection coefficients can be listed as follows

$$\begin{aligned} SH &\to SH, \quad \text{no critical incidence.} \\ P &\to P \quad \text{and} \quad P \to SV, \text{no critical incidence.} \\ SV &\to SV \quad \text{and} \quad SV \to P, \quad \text{critical incidence can occur.} \end{aligned}$$

Apparently, the procedure described above can be extended to other boundary conditions almost straightforward, though it will require bulkier calculations.

7.4 Basic equations of the ray method in elastodynamics

Similarly to the wave equation, we can develop the ray theory in elastodynamics as an extension to the theory of plane wave propagation in homogeneous medium applied to a heterogeneous but slowly varying one. In the latter case we suppose that the characteristics of the elastic medium remain to be almost constant on the wavelength interval in all directions. Apparently, this has a heuristic rather than precise mathematical sense, but it allows us to understand more clearly the calculations necessary for deriving the basic equations of the ray theory.

We consider below an inhomogeneous but isotropic medium. The elastic parameters ρ, λ, μ are supposed to be smooth functions of the coordinates.

The elastodynamic equations for the displacement vector $\vec{U}$ read

$$(\lambda+\mu)\frac{\partial^2 U_j}{\partial x_i \partial x_j}+\mu\frac{\partial^2 U_i}{\partial x_j \partial x_j}+\frac{\partial \lambda}{\partial x_i}\frac{\partial U_j}{\partial x_j}+\frac{\partial \mu}{\partial x_j}\left(\frac{\partial U_i}{\partial x_j}+\frac{\partial U_j}{\partial x_i}\right)=\rho\frac{\partial^2 U_i}{\partial t^2}\,, \quad i=1,2,3. \tag{7.22}$$

If we take into account that

$$\left[\nabla\mu, \mathrm{rot}\,\vec{U}\right]_i=\frac{\partial \mu}{\partial x_j}\frac{\partial U_j}{\partial x_i}-\frac{\partial \mu}{\partial x_j}\frac{\partial U_i}{\partial x_j}$$

then we can present the equations in the following vectorial form

$$(\lambda+\mu)\nabla\,\mathrm{div}\,\vec{U}+\mu\Delta\vec{U}+\nabla\lambda\,\mathrm{div}\,\vec{U}+[\nabla\mu,\mathrm{rot}\,\vec{U}]+2(\nabla\mu,\nabla)\vec{U}=\rho\frac{\partial^2\vec{U}}{\partial t^2}\,. \tag{7.23}$$

In the case of a harmonic in time wave field, or in the case of the ray method in the frequency domain, we seek a solution of the elastodynamic equations in the form

$$\vec{U}=\vec{A}e^{-i\omega(t-\tau)}\,, \quad i^2=-1\,, \tag{7.24}$$

where $\vec{A}$ is the amplitude and τ is the eikonal.

Unlike the case of homogeneous media we cannot expect the amplitude $\vec{A}$ and the eikonal τ to be a constant and a linear function of the arguments x_j, $j=1,2,3$, respectively. Therefore for derivatives of the displacement vector (7.24) we get

$$\begin{aligned}\frac{\partial U_k}{\partial x_j} &= \frac{\partial}{\partial x_j}A_k e^{-i\omega(t-\tau)}=e^{-i\omega(t-\tau)}\left[i\omega\frac{\partial\tau}{\partial x_j}A_k+\frac{\partial A_k}{\partial x_j}\right]\,,\\ \frac{\partial^2 U_k}{\partial x_j\partial x_l} &= e^{-i\omega(t-\tau)}\left[-\omega^2\frac{\partial\tau}{\partial x_l}\frac{\partial\tau}{\partial x_j}A_k+i\omega\left(\frac{\partial\tau}{\partial x_l}\frac{\partial A_k}{\partial x_j}+\frac{\partial\tau}{\partial x_j}\frac{\partial A_k}{\partial x_l}+\right.\right.\\ &\qquad\left.\left.+\frac{\partial^2\tau}{\partial x_l\partial x_j}A_k\right)+\frac{\partial^2 A_k}{\partial x_l\partial x_j}\right]\,.\end{aligned}$$

By inserting expression (7.24) into the elastodynamic equations (7.22) and by canceling the exponents, we arrive at the following equality

$$\begin{aligned}&-\omega^2\left\{(\lambda+\mu)\frac{\partial\tau}{\partial x_k}\frac{\partial\tau}{\partial x_j}A_j+\mu\frac{\partial\tau}{\partial x_j}\frac{\partial\tau}{\partial x_j}A_k\right\}+\\ &+i\omega\left\{(\lambda+\mu)\left(\frac{\partial\tau}{\partial x_k}\frac{\partial A_j}{\partial x_j}+\frac{\partial\tau}{\partial x_j}\frac{\partial A_j}{\partial x_k}+\frac{\partial^2\tau}{\partial x_k\partial x_j}A_j\right)+\right.\\ &\qquad+\mu\left(2\frac{\partial\tau}{\partial x_j}\frac{\partial A_k}{\partial x_j}+\frac{\partial^2\tau}{\partial x_j\partial x_j}A_k\right)+\frac{\partial\lambda}{\partial x_k}\frac{\partial\tau}{\partial x_j}A_j+\\ &\qquad\left.+\frac{\partial\mu}{\partial x_j}\left(\frac{\partial\tau}{\partial x_j}A_k+\frac{\partial\tau}{\partial x_k}A_j\right)\right\}+\\ &+\left\{(\lambda+\mu)\frac{\partial^2 A_j}{\partial x_k\partial x_j}+\mu\frac{\partial^2 A_k}{\partial x_j\partial x_j}+\frac{\partial\lambda}{\partial x_k}\frac{\partial A_j}{\partial x_j}+\frac{\partial\mu}{\partial x_j}\left(\frac{\partial A_k}{\partial x_j}+\frac{\partial A_j}{\partial x_k}\right)\right\}=\\ &\qquad=-\omega^2\rho A_k\,, \quad k=1,2,3\,.\end{aligned}$$

By gathering the terms of the same order with respect to ω, we can present the latter equations as follows

$$-\omega^2 N_k(\vec{A}) + i\omega M_k(\vec{A}) + L_k(\vec{A}) = 0\,, \quad k = 1,2,3, \tag{7.25}$$

where

$$\begin{aligned}
N_k &= (\lambda+\mu)\frac{\partial\tau}{\partial x_k}\frac{\partial\tau}{\partial x_j}A_j + \left[\mu\left(\frac{\partial\tau}{\partial x_j}\frac{\partial\tau}{\partial x_j}\right) - \rho\right]A_k\,,\\
M_k(\vec{A}) &= (\lambda+\mu)\left(\frac{\partial\tau}{\partial x_k}\frac{\partial A_j}{\partial x_j} + \frac{\partial\tau}{\partial x_j}\frac{\partial A_j}{\partial x_k} + \frac{\partial^2\tau}{\partial x_k\partial x_j}A_j\right) + \\
&\quad + \mu\left(2\frac{\partial\tau}{\partial x_j}\frac{\partial A_k}{\partial x_j} + \frac{\partial^2\tau}{\partial x_j\partial x_j}A_k\right) + \frac{\partial\lambda}{\partial x_k}\frac{\partial\tau}{\partial x_j}A_j + \\
&\quad + \frac{\partial\mu}{\partial x_j}\left(\frac{\partial\tau}{\partial x_j}A_k + \frac{\partial\tau}{\partial x_k}A_j\right)\,,
\end{aligned}$$

and

$$L_k = (\lambda+\mu)\frac{\partial^2 A_j}{\partial x_k\partial x_j} + \mu\frac{\partial^2 A_k}{\partial x_j\partial x_j} + \frac{\partial\lambda}{\partial x_k}\frac{\partial A_j}{\partial x_j} + \frac{\partial\mu}{\partial x_j}\left(\frac{\partial A_k}{\partial x_j} + \frac{\partial A_j}{\partial x_k}\right)\,.$$

Next, let us present the amplitude $\vec{A}$ in equation (7.24) as an asymptotic series

$$\vec{A} = \sum_{n=0}^{\infty}\frac{\vec{u}_n}{(i\omega)^n} \tag{7.26}$$

and insert it into equations (7.25) for the amplitude. Due to the fact that N_k, M_k, L_k, $k = 1,2,3$, in equation (7.25) are linear operators we can present the final result in the following form

$$\sum_{m=-2}^{\infty}\frac{N_k(\vec{u}_{m+2})}{(i\omega)^m} + \sum_{m=-1}^{\infty}\frac{M_k(\vec{u}_{m+1})}{(i\omega)^m} + \sum_{m=0}^{\infty}\frac{L_k(\vec{u}_m)}{(i\omega)^m} = 0\,. \tag{7.27}$$

Indeed, for example, the corresponding calculations for the operator N_k are the following

$$\begin{aligned}
-\omega^2 N_k\left(\sum_{n=0}^{\infty}\frac{\vec{u}_n}{(i\omega)^n}\right) &= \sum_{n=0}^{\infty}(i\omega)^2\frac{N_k(\vec{u}_n)}{(i\omega)^n} = \\
= \sum_{n=0}^{\infty}\frac{N_k(\vec{u}_n)}{(i\omega)^{n-2}} &= \sum_{m=-2}^{\infty}\frac{N_k(\vec{u}_{m+2})}{(i\omega)^m}\,.
\end{aligned}$$

By collecting terms with equal powers of $i\omega$ in equation (7.27) and by equating them to zero we arrive at the recurrent system of equations

$$\begin{aligned}
&N_k(\vec{u}_o) = 0\,,\\
&N_k(\vec{u}_1) + M_k(\vec{u}_o) = 0\,,\\
&N_k(\vec{u}_{m+2}) + M_k(\vec{u}_{m+1}) + L_k(\vec{u}_m) = 0\,, \quad m = 0,1,2,\dots\,, \quad k = 1,2,3.
\end{aligned} \tag{7.28}$$

In principle, equations (7.28) can be solved step by step and we can construct an arbitrary finite number of terms in the asymptotic series (7.26). Unfortunately, corresponding formulas for $\vec{u}_m$ are given implicitly and that causes difficulties in their applications. It should also be noted that the mathematical procedure for solving equations (7.28) possesses the following specific property. The first equation in (7.28) is actually a homogeneous system of linear algebraic equations with respect to the coordinates of vector $\vec{u}_o$ and we look for a nontrivial solution of it. Suppose it is solved. The second equation then is already a non-homogeneous system and we have to solve it in this case when the corresponding homogeneous system has a non-zero solution. But it is known in linear algebra that the necessary and sufficient condition for this reads (in case the main matrix of the system is symmetrical): the right-hand side of the non-homogeneous system must be orthogonal to the solutions of the corresponding homogeneous system. It is precisely this condition that gives rise to the transport equation. This property summarizes all successive equations in (7.28).

7.5 The eikonal equation

Consider first the following equation

$$N_k(\vec{u}_o) = 0\,, \quad k = 1, 2, 3. \tag{7.29}$$

Being a linear system of homogeneous algebraic equations, it can be presented in the matrix form

$$\mathbf{N}\vec{u}_o = 0\,,$$

where $\mathbf{N}$ is the following 3x3 matrix

$$N_{kj} = (\lambda + \mu)\frac{\partial \tau}{\partial x_k}\frac{\partial \tau}{\partial x_j} + \big(\mu(\nabla\tau)^2 - \rho\big)\delta_{kj}\,.$$

Hence, equations (7.29) will have a non-zero solution if and only if $\det \mathbf{N} = 0$. By denoting by Λ the following expression $\Lambda = \mu(\nabla\tau)^2 - \rho$ we get (compare with equations (7.4)!)

$$\det \mathbf{N} = \begin{vmatrix} (\lambda+\mu)\frac{\partial \tau}{\partial x_1}\frac{\partial \tau}{\partial x_1} + \Lambda & (\lambda+\mu)\frac{\partial \tau}{\partial x_1}\frac{\partial \tau}{\partial x_2} & (\lambda+\mu)\frac{\partial \tau}{\partial x_1}\frac{\partial \tau}{\partial x_3} \\ (\lambda+\mu)\frac{\partial \tau}{\partial x_2}\frac{\partial \tau}{\partial x_1} & (\lambda+\mu)\frac{\partial \tau}{\partial x_2}\frac{\partial \tau}{\partial x_2} + \Lambda & (\lambda+\mu)\frac{\partial \tau}{\partial x_2}\frac{\partial \tau}{\partial x_3} \\ (\lambda+\mu)\frac{\partial \tau}{\partial x_3}\tfrac{\partial \tau}{\partial x_1} & (\lambda+\mu)\frac{\partial \tau}{\partial x_3}\frac{\partial \tau}{\partial x_2} & (\lambda+\mu)\frac{\partial \tau}{\partial x_3}\frac{\partial \tau}{\partial x_3} + \Lambda \end{vmatrix} =$$

$$= \Lambda^3 + \Lambda^2\left[(\lambda+\mu)\left(\frac{\partial \tau}{\partial x_1}\right)^2 + (\lambda+\mu)\left(\frac{\partial \tau}{\partial x_2}\right)^2 + (\lambda+\mu)\left(\frac{\partial \tau}{\partial x_3}\right)^2\right] =$$

$$= \Lambda^2(\Lambda + (\lambda+\mu)(\nabla\tau)^2)\,.$$

Thus, equation $\det \mathbf{N} = 0$ leads to the following possibilities.

1) Compressional waves:

$$\Lambda + (\lambda+\mu)(\nabla\tau)^2 = (\lambda+2\mu)(\nabla\tau)^2 - \rho = 0$$

and

$$(\nabla\tau)^2 = \frac{1}{a^2}\,, \quad a^2 = \frac{\lambda+2\mu}{\rho}\,. \tag{7.30}$$

2) Shear waves:

$$\Lambda = \mu(\nabla\tau)^2 - \rho = 0 \Rightarrow (\nabla\tau)^2 = \frac{1}{b^2}\,, \quad b^2 = \frac{\mu}{\rho}\,. \tag{7.31}$$

Thus, similarly to the case of plane waves in homogeneous media we get two velocities a and b for compressional and shear waves, respectively. But unlike that case, in inhomogeneous media we obtain a partial differential equation (7.30) or (7.31) for the eikonal τ.

7.6 Transport equations

Transport equations for the main term $\vec{u}_o$ of the ray series arise from the second equations in (7.28)

$$N_k(\vec{u}_1) = -M_k(\vec{u}_o)\,, \quad k = 1,2,3. \tag{7.32}$$

Further, it is convenient to write down operator M_k in the vectorial form. If we put, as usual, $\vec{M}(\vec{A}) = M_k(\vec{A})\vec{i}_k$, where $\vec{i}_k$, $k = 1,2,3$ are the basis vectors of the Cartesian coordinates, then after some mathematics we obtain

$$\begin{aligned}\vec{M}(\vec{A}) &= (\lambda+\mu)\big\{\nabla\tau \text{ div } \vec{A} + \nabla(\nabla\tau,\vec{A})\big\} + \mu\big[2(\nabla\tau,\nabla)\vec{A} + \Delta\tau\vec{A}\big] + \\ &\quad + \nabla\lambda(\nabla\tau,\vec{A}) + (\nabla\mu,\nabla\tau)\vec{A} + (\nabla\mu,\vec{A})\nabla\tau\,,\end{aligned} \tag{7.33}$$

or

$$\begin{aligned}\vec{M}(\vec{A}) &= (\lambda+\mu)\big[\nabla\tau \text{ div } \vec{A} + \nabla(\vec{A},\nabla\tau)\big] + \mu\Delta\tau\vec{A} + 2\mu(\nabla\tau,\nabla)\vec{A} + \\ &\quad + \nabla\lambda(\nabla\tau,\vec{A}) + [\nabla\mu,[\nabla\tau,\vec{A}]] + 2(\nabla\mu,\nabla\tau)\vec{A}\,.\end{aligned} \tag{7.34}$$

Compressional waves.

In this case we may seek the amplitude $\vec{A}$ in the form

$$\vec{A} \equiv \vec{u}_o = \varphi_o a \nabla\tau\,, \tag{7.35}$$

where $a = \sqrt{\dfrac{\lambda+2\mu}{\rho}}$ is the velocity of P-waves.

Further, we need to use the following auxiliary formulas:

$$\begin{aligned}\text{div } \vec{A} &= \varphi_o a\Delta\tau + \varphi_o(\nabla a,\nabla\tau) + a(\nabla\varphi_o,\nabla\tau)\,, \\ \nabla(\nabla\tau,\vec{A}) &= \nabla\left(\frac{\varphi_o}{a}\right)\,, \\ (\nabla\tau,\nabla)\vec{A} &= a(\nabla\tau,\nabla\varphi_o)\nabla\tau + \varphi_o(\nabla\tau,\nabla a)\nabla\tau + \frac{1}{2}\varphi_o a\nabla\left(\frac{1}{a^2}\right)\,.\end{aligned}$$

One can verify them by inserting equation (7.35) for amplitude $\vec{A}$ into the left-hand sides of the latter equations.

Now let us check that $\vec{u}_o$ given by formula (7.35) satisfies the homogeneous equations $\mathbf{N}\vec{u}_o = 0$. Indeed, by inserting $\vec{u}_o$ into equations (7.29) we consistently obtain

$$
\begin{aligned}
N_k(\vec{u}_o) &= \varphi_o a \sum_{n=1}^{3} \left[(\lambda+\mu)\frac{\partial\tau}{\partial x_k}\frac{\partial\tau}{\partial x_n} + \Lambda\delta_{kn}\right]\frac{\partial\tau}{\partial x_n} = \\
&= \varphi_o a \left[(\lambda+\mu)\frac{\partial\tau}{\partial x_k}(\nabla\tau)^2 + \Lambda\frac{\partial\tau}{\partial x_k}\right] = \\
&= \varphi_o a \frac{\partial\tau}{\partial x_k}\left[(\lambda+\mu)(\nabla\tau)^2 + \Lambda\right] = 0, \quad k = 1,2,3,
\end{aligned}
$$

due to $\Lambda = -(\lambda+\mu)(\nabla\tau)^2$ in the case under consideration.

Therefore for that non-homogeneous system of equations (7.32) admitting a solution with respect to $\vec{u}_1$, we must impose the following equation of orthogonality $(\vec{M}(\vec{u}_o), \vec{u}_o) = 0$, which is equivalent to the following

$$
\big(\vec{M}(\vec{u}_o), \nabla\tau\big) = 0\,. \tag{7.36}
$$

Then, by inserting the amplitude $\vec{u}_o$ in the form (7.35) into equation (7.34) and multiplying it by $\nabla\tau$, we get

$$
\begin{aligned}
\big(\vec{M}(\vec{u}_o), \nabla\tau\big) = (\lambda+\mu)\, &\big[(\nabla\tau)^2\{\varphi_o a\Delta\tau + \varphi_o(\nabla a, \nabla\tau) + a(\nabla\varphi_o, \nabla\tau)\} \\
&+ \left(\nabla\frac{\varphi_o}{a}, \nabla\tau\right)\Big] + \mu\Delta\tau\varphi_o a(\nabla\tau)^2 + \\
&+ 2\mu\left[a(\nabla\tau, \nabla\varphi_o)(\nabla\tau)^2 + \varphi_o(\nabla\tau, \nabla a)(\nabla\tau)^2 + \frac{1}{2}\varphi_o a\left(\nabla\frac{1}{a^2}, \nabla\tau\right)\right] + \\
&+ (\nabla\lambda, \nabla\tau)\varphi_o a(\nabla\tau)^2 + 2(\nabla\mu, \nabla\tau)\varphi_o a(\nabla\tau)^2\,.
\end{aligned}
$$

Finally, taking into account that $(\nabla\tau)^2 = 1/a^2$ we obtain the following result

$$
\begin{aligned}
\big(\vec{M}(\vec{u}_o), \nabla\tau\big) &= (\lambda+2\mu)\frac{\varphi_o}{a}\Delta\tau + \frac{2}{a}(\nabla\varphi_o, \nabla\tau)(\lambda+2\mu) + \frac{\varphi_o}{a}\big(\nabla\tau, \nabla(\lambda+2\mu)\big) = \\
&= \frac{1}{a}\big\{2(\nabla\tau, \nabla\varphi_o)a^2\rho + a^2\rho\varphi_o\Delta\tau + \varphi_o(\nabla\tau, \nabla a^2\rho)\big\}
\end{aligned}
$$

due to $\lambda + 2\mu = a^2\rho$.

Now the transport equation for φ_o follows from (7.36) and takes the form

$$
2(\nabla\tau, \nabla\varphi_o)a^2\rho + a^2\rho\varphi_o\Delta\tau + \varphi_o(\nabla\tau, \nabla a^2\rho) = 0\,. \tag{7.37}
$$

As a final step let us seek a scalar amplitude φ_o in the form

$$
\varphi_o = \frac{1}{\sqrt{a^2\rho}}\tilde{\varphi}_o\,.
$$

Then we get

$$\nabla\varphi_o = \frac{1}{\sqrt{a^2\rho}}\nabla\tilde{\varphi}_o - \frac{1}{2}\frac{\tilde{\varphi}_o}{(a^2\rho)^{3/2}}\nabla a^2\rho$$

and by inserting the latter formula in equation (7.37) we obtain

$$2\sqrt{a^2\rho}\,(\nabla\tau,\nabla\tilde{\varphi}_o) - \frac{\tilde{\varphi}_o}{\sqrt{a^2\rho}}(\nabla a^2\rho,\nabla\tau) + \sqrt{a^2\rho}\,\tilde{\varphi}_o\Delta\tau + \frac{\tilde{\varphi}_o}{\sqrt{a^2\rho}}(\nabla\tau,\nabla a^2\rho) = 0\,.$$

Now it follows from the above that $\tilde{\varphi}_o$ has to be a solution of the equation

$$2(\nabla\tau,\nabla\tilde{\varphi}_o) + \tilde{\varphi}_o\Delta\tau = 0\,, \qquad (7.38)$$

which is exactly the same as we had in the case of the wave equation!

We know that a solution of the transport equation (7.38) can be written in the form

$$\tilde{\varphi}_o = \frac{\psi_o(\alpha,\beta)}{\sqrt{\frac{1}{a}J}}\,, \quad J = \frac{1}{a}\left|\frac{D(x_1,x_2,x_3)}{D(\tau,\alpha,\beta)}\right|\,,$$

where τ,α,β are the ray coordinates and J is the geometrical spreading of a ray tube.

Shear waves.

Let us seek amplitude $\vec{u}_o$ in the form

$$\vec{u}_o = B\vec{g}^{\,(1)} + C\vec{g}^{\,(2)}\,, \qquad (7.39)$$

where B and C are some unknown scalar functions and $\vec{g}^{\,(k)}, k = 1,2$, are mutually orthogonal unit vectors. Note that both of them are orthogonal to $\nabla\tau$, i.e.

$$\big(\nabla\tau,\vec{g}^{\,(k)}\big) = 0\,, \quad k = 1,2. \qquad (7.40)$$

It follows immediately from here that both $B\vec{g}^{\,(1)}$ and $C\vec{g}^{\,(2)}$ satisfy the homogeneous equations (7.28). Indeed, in the case under consideration $\Lambda = 0$ and therefore

$$N_k\big(\vec{g}^{\,(j)}\big) = (\lambda+\mu)\frac{\partial\tau}{\partial x_k}\big(\nabla\tau,\vec{g}^{\,(j)}\big) = 0\,, \quad k = 1,2,3,\; j = 1,2.$$

This time we must impose two conditions of orthogonality $\big(\vec{M}(\vec{u}_o),\vec{g}^{\,(k)}\big) = 0$, $k = 1,2$, in order to make the non-homogeneous system (7.32) a solvable one with respect to $\vec{u}_1$.

By using formula (7.33) for $\vec{M}(\vec{A})$ and multiplying it by $\vec{g}^{\,(k)}$ we get

$$\big(\vec{M}(\vec{A}),\vec{g}^{\,(k)}\big) = 2\mu\big((\nabla\tau,\nabla)\vec{A},\vec{g}^{\,(k)}\big) + \Delta\tau\big(\vec{A},\vec{g}^{\,(k)}\big) + \big(\nabla\mu,\nabla\tau\big)\big(\vec{A},\vec{g}^{\,(k)}\big)\,.$$

Then by inserting $\vec{A} = \vec{u}_o$ into the latter equations we obtain

$$\begin{aligned}\big(\vec{M}(\vec{A}),\vec{g}^{\,(1)}\big) \;=\;& 2\mu(\nabla\tau,\nabla B) + \Delta\tau B\mu + (\nabla\tau,\nabla\mu)B + \\ & + 2\mu\big[B\big((\nabla\tau,\nabla)\vec{g}^{\,(1)},\vec{g}^{\,(1)}\big) + C\big((\nabla\tau,\nabla)\vec{g}^{\,(2)},\vec{g}^{\,(1)}\big)\big]\,,\end{aligned} \qquad (7.41)$$

and

$$\big(\vec{M}(\vec{A}),\vec{g}^{\,(2)}\big) = 2\mu(\nabla\tau,\nabla C)+\mu\Delta\tau C+(\nabla\mu,\nabla\tau)C+ + 2\mu\big[B\big((\nabla\tau,\nabla)\vec{g}^{\,(1)},\vec{g}^{\,(2)}\big)+C\big((\nabla\tau,\nabla)\vec{g}^{\,(2)},\vec{g}^{\,(2)}\big)\big]\,. \tag{7.42}$$

Let us consider in more details the following operator

$$(\nabla\tau,\nabla)=\frac{\partial\tau}{\partial x_j}\frac{\partial}{\partial x_j}\,.$$

If $\vec{t}$ is a tangent vector to a ray of unit length and s is the arc length along the ray, then we have

$$(\nabla\tau,\nabla)=\left(\frac{\vec{t}}{b},\nabla\right)=\frac{1}{b}(\vec{t},\nabla)=\frac{1}{b}\frac{d}{ds}\,,$$

where b is the shear wave velocity and d/ds is derivative along the ray. Hence, in equations (7.41) and (7.42) derivatives of vectors $\vec{g}^{\,(k)}, k=1,2$, along the ray appear, i.e. we have to calculate $(d/ds)\vec{g}^{\,(k)}, k=1,2$.

Suppose, we choose vectors $\vec{g}^{\,(k)}$ as follows

$$\vec{g}^{\,(1)}=\vec{e}_1\,,\quad \vec{g}^{\,(2)}=\vec{e}_2\,,$$

where $\vec{e}_1,\vec{e}_2$ are unit vectors of the ray centered coordinates. In this case

$$\frac{d}{ds}\vec{g}^{\,(m)}=\frac{d}{ds}\vec{e}_m=\kappa_m\vec{t},\quad \kappa_m=\frac{1}{b}\frac{\partial b}{\partial q_m}\bigg|_{q_1=q_2=0}\,, m=1,2,$$

and formulas (7.41) and (7.42) take the form

$$\big(\vec{M}(\vec{A}),\vec{g}^{\,(1)}\big) = \big(\vec{M}(\vec{A}),\vec{e}_1\big)=2\mu(\nabla\tau,\nabla B)+\mu B\Delta\tau+(\nabla\tau,\nabla\mu)B\,,$$

$$\big(\vec{M}(\vec{A}),\vec{e}_2\big) = 2\mu(\nabla\tau,\nabla C)+\mu C\Delta\tau+C(\nabla\tau,\nabla\mu)\,,$$

and we obtain two independent transport equations of the same type

$$2(\nabla\tau,\nabla B)+B\Delta\tau+(\nabla\tau,\nabla\ln\mu)B=0\,,$$
$$2(\nabla\tau,\nabla C)+C\Delta\tau+(\nabla\tau,\nabla\ln\mu)C=0\,.$$

By substituting B and C in the latter equations by

$$B=\frac{\tilde{B}}{\sqrt{\mu}}\quad\text{and } C=\frac{\tilde{C}}{\sqrt{\mu}}\quad(\mu=\rho b^2)$$

we obtain the transport equations which coincide with the transport equation derived in Chapter 1 for the scalar wave field

$$\begin{aligned}&2(\nabla\tau,\nabla\tilde{B})+\tilde{B}\Delta\tau=0\,,\\&2(\nabla\tau,\nabla\tilde{C})+\tilde{C}\Delta\tau=0\,.\end{aligned}\tag{7.43}$$

Finally, we obtain the following formulas for $\tilde{B}$ and $\tilde{C}$

$$\tilde{B} = \frac{\psi_o^{(1)}(\alpha,\beta)}{\sqrt{\frac{1}{b}J}}\,, \quad \tilde{C} = \frac{\psi_o^{(2)}(\alpha,\beta)}{\sqrt{\frac{1}{b}J}}\,, \quad J = \frac{1}{b}\left|\frac{D(x_1,x_2,x_3)}{D(\tau,\alpha,\beta)}\right|\,, \tag{7.44}$$

where, in general, functions $\psi_o^{(1)}$ and $\psi_o^{(2)}$ are different.

Summary

In elastodynamics we have a family of P - rays propagating with velocity $a = \sqrt{(\lambda+2\mu)/\rho}$. The corresponding ray - method formula for P wave reads

$$\vec{U} = \frac{\psi_o(\alpha,\beta)}{\sqrt{(1/a)J}}\frac{e^{i\omega(\tau-t)}}{\sqrt{a^2\rho}}a\nabla\tau\,. \tag{7.45}$$

Then, we also have a family of S - rays propagating with velocity $b = \sqrt{\mu/\rho}$ and two shear waves

$$\begin{aligned} \vec{U}^{(1)} &= \frac{\psi_o^{(1)}(\alpha,\beta)}{\sqrt{(1/b)J}\sqrt{b^2\rho}}e^{i\omega(\tau-t)}\vec{e}_1\,, \\ \vec{U}^{(2)} &= \frac{\psi_o^{(2)}(\alpha,\beta)}{\sqrt{(1/b)J}\sqrt{b^2\rho}}e^{i\omega(\tau-t)}\vec{e}_2\,. \end{aligned} \tag{7.46}$$

Note that vectors $\vec{e}_1$, $\vec{e}_2$ of the ray centered coordinates seem to be more convenient for describing polarization of the shear waves then the normal and binormal to the ray, because the polarization vectors rotate with respect to the normal and binormal while propagating along the ray. Apparently, we can construct a shear wave of arbitrary polarization as a linear combination of $\vec{U}^{(1)}$ and $\vec{U}^{(2)}$ in equations (7.46).

In order to apply formulas (7.45) and (7.46) we have to find appropriate values for the functions ψ_o and here we face the problem of initial data for the ray series. Within the frames of the leading (or main) term of the ray series this problem can be solved quite easily by means of matching of asymptotics. But this requires rather bulky mathematics even for the second term of the series in case of a point source. For that matter see, for example, Babich and Kirpichnikova (1979).

7.7 Point sources; initial data for P and S waves

Consider now two types of point sources: the center of dilatation and the center of rotation, which are described by the following formulas, respectively,

$$\vec{F} = -\nabla\delta(M-M_o) \quad \text{and} \quad \vec{F} = -\text{rot}\,(\vec{l}\,\delta(M-M_o))\,, \tag{7.47}$$

where $\vec{l}$ is a constant vector of unit length. Both of the sources are widely used in theoretical geophysics.

Initial data, i.e. formulas for $\psi_o, \psi_o^{(1)}, \psi_o^{(2)}$, can be obtained by matching the main terms of the ray series in a vicinity of the sources and the exact solutions of corresponding problems for a homogeneous isotropic medium.

As a first step, we have to freeze the density ρ and Lame's parameters λ, μ by their values ρ_o, λ_o, μ_o at the source point M_o. Then we arrive at the elastodynamic equations for a homogeneous isotropic medium

$$(\lambda_o + \mu_o)\nabla(\text{div } \vec{U}) + \mu_o \Delta \vec{U} + \vec{f} = \rho_o \frac{\partial^2 \vec{U}}{\partial t^2} . \tag{7.48}$$

Equation (7.48) can be rewritten in an equivalent form

$$(\lambda_o + 2\mu_o)\nabla(\text{div } \vec{U}) - \mu_o \text{rot}(\text{rot } \vec{U}) + \vec{f} = \rho_o \frac{\partial^2 \vec{U}}{\partial t^2} \tag{7.49}$$

if the following identity is used

$$\Delta \vec{U} = \nabla(\text{div } \vec{U}) - \text{rot}(\text{rot } \vec{U}) .$$

In order to obtain appropriate problems for equations (7.48) or (7.49) we have to consider the harmonic in time wave field and substitute the density of body forces $\vec{f}$ by one of the point sources (7.47).

The center of dilatation.

In this case we have

$$\vec{f} = \nabla \delta(M - M_o) \tag{7.50}$$

and seek a solution of equation (7.49) in the form

$$\vec{U} = e^{-i\omega t} \nabla G , \tag{7.51}$$

where G is a scalar function.

By inserting formulas (7.50) and (7.51) into (7.49) and taking into account the well-known identity rot $\nabla G \equiv 0$ we obtain the following equation for the function G

$$\nabla \big\{ (\lambda_o + 2\mu_o) \text{ div grad } G + \rho_o \omega^2 G \big\} = -\nabla \delta(M - M_o) . \tag{7.52}$$

Now this can be reduced to the point source problem for Helmholtz equation

$$\Delta G + \frac{\omega^2}{a_o^2} G = -\frac{1}{\lambda_o + 2\mu_o} \delta(M - M_o) , \tag{7.53}$$

where $a_o^2 = (\lambda_o + 2\mu_o)/\rho_o$. Note that Sommerfeld's radiation conditions have to be added to equation (7.53) so that an appropriate unique solution can be obtained. The radiation conditions mean that G must contain only the wave emanating from the source, with a suitable decline of its amplitude with an increasing distance to the source. Thus, we finally obtain from (7.53) that

$$G = \frac{1}{4\pi R(\lambda_o + 2\mu_o)} \exp\left\{ i\omega \frac{R}{a_o} \right\} , \tag{7.54}$$

where $R = \sqrt{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2}$. By inserting formula (7.54) into (7.51) we obtain a desirable result for the displacement vector $\vec{U}$

$$\vec{U} = e^{-i\omega(t-R/a_o)} \frac{\omega}{4\pi a_o^2 \rho_o} \left(\frac{i}{a_o R} - \frac{1}{\omega R^2} \right) \nabla R \,. \tag{7.55}$$

Due to $\nabla R = \vec{R}/|\vec{R}|$, where $\vec{R} = (x-x_o)\vec{i} + (y-y_o)\vec{j} + (z-z_o)\vec{k}$ being the radius-vector, formula (7.55) describes the compressional wave emanating from the source. It has the form of a ray series, but it consists only of two terms. The main term declines as R^{-1} and the second one as R^{-2}.

Now let us come back to the ray-method formula (7.45) for P-wave. In a small vicinity of the source, the rays are almost straight lines, therefore in the first approximation we have

$$\tau \sim \frac{R}{a_o} \,, \quad a_o \nabla \tau \sim \frac{\vec{R}}{|\vec{R}|} \,. \tag{7.56}$$

Let angles θ and ϕ $(0 \le \theta \le \pi \,; \quad 0 \le \phi < 2\pi)$ of the spherical coordinates be the ray parameters, then for the geometrical spreading J we have

$$J \sim s^2 \sin\theta = R^2 \sin\theta \,. \tag{7.57}$$

By inserting approximate formulas (7.56) and (7.57) into equation (7.45), we get

$$\vec{U} \simeq \frac{\psi_o(\theta,\phi)}{\sqrt{(1/a_o) R^2 \sin\theta}} \frac{\exp\left[-i\omega\left(t - \frac{R}{a_o}\right)\right]}{\sqrt{a_o^2 \rho_o}} \frac{\vec{R}}{|\vec{R}|} \,.$$

By comparing the latter formula with the first item in equation (7.55) we obtain the following expression for $\psi_o(\theta,\phi)$

$$\psi_o = \frac{i\omega}{4\pi} \frac{\sqrt{\sin\theta}}{a_o^{5/2} \sqrt{\rho_o}} \,. \tag{7.58}$$

Thus, the insertion of formula (7.58) into equation (7.45) completes the ray-method expression for the displacement vector $\vec{U}$ in case of the center of dilatation which can be used on an arbitrary but finite distance from the source.

The center of rotation.

Now we have

$$\vec{f} = \text{rot}\left(\vec{l}\,\delta(M - M_o)\right) \tag{7.59}$$

and seek a solution of elastodynamic equation (7.49) in the form

$$\vec{U} = e^{-i\omega t} \text{rot}\,\vec{\psi} \tag{7.60}$$

with an unknown vector-function $\vec{\psi}$. By inserting both (7.59) and (7.60) into equation (7.49) we obtain consistently

$$\text{rot}\,\{\mu_o \Delta\vec{\psi} + \rho_o \omega^2 \vec{\psi}\} = -\text{rot}\left(\vec{l}\,\delta(M - M_o)\right) \,,$$

and

$$\Delta\vec{\psi} + \frac{\omega^2}{b_o^2}\vec{\psi} = -\frac{1}{\mu_o}\vec{l}\,\delta(M - M_o)\,, \quad b_o^2 = \mu_o/\rho_o\,. \tag{7.61}$$

Taking into account Sommerfeld's radiation conditions (they must hold in this case too!) we get from (7.61)

$$\vec{\psi} = \frac{\vec{l}}{4\pi\mu_o R}\exp\{i\omega R/b_o\}\,. \tag{7.62}$$

Due to

$$\text{rot}\,\vec{\psi} = \frac{1}{4\pi\mu_o}\left[\text{grad}\frac{e^{i\omega R/b_o}}{R}\,,\,\vec{l}\,\right] = \frac{\omega}{4\pi\mu_o}\left(\frac{i}{b_o R} - \frac{1}{\omega R^2}\right)[\nabla R, \vec{l}\,]e^{i\omega R/b_o}$$

we obtain the following expression for the displacement vector (7.60)

$$\vec{U} = \frac{\omega}{4\pi\mu_o}\left(\frac{i}{b_o R} - \frac{1}{\omega R^2}\right)[\nabla R, \vec{l}\,]\exp\,[i\omega(R/b_o - t)]\,. \tag{7.63}$$

Note that the vector product $[\nabla R, \vec{l}]$ vanishes if ∇R is collinear with the vector $\vec{l}$. This means that the center of rotation does not irradiate along a straight line fixed by the vector $\vec{l}$. Polarization of the displacement vector $\vec{U}$ in (7.63) is orthogonal to $\nabla R = \vec{R}/|\vec{R}|$.

Consider further the ray-method formulas (7.46) in a vicinity of the point source. Apparently, equations (7.56) and (7.57) hold true in the case under consideration if a_o is replaced by the velocity b_o. Let us set an initial orientation of the unit vector $\vec{e}_1$ at the source point $s = 0$ by the formula

$$\vec{e}_1\big|_{s=0} = \frac{[\nabla\vec{R}, \vec{l}\,]}{\big|[\nabla R, \vec{l}\,]\big|} = \left[\frac{\vec{R}}{|\vec{R}|}, \vec{l}\,\right]\frac{1}{\big|[\nabla R, \vec{l}\,]\big|} \tag{7.64}$$

which is correctly defined when $[\nabla R, \vec{l}\,] \neq 0$.

Then $\vec{U}^{(1)}$ in equations (7.46) can be rewritten in the form

$$\vec{U}^{(1)} \sim \frac{\psi_o^{(1)}(\theta,\phi)}{\sqrt{(1/b_o)R^2\sin\theta}\sqrt{b_o^2\rho_o}}\frac{[\nabla R, \vec{l}\,]}{|[\nabla R, \vec{l}\,]|}e^{i\omega(R/b_o - t)}\,. \tag{7.65}$$

By comparing now expression (7.65) and the first term in equation (7.63) we obtain the final result

$$\psi_o^{(1)}(\theta,\phi) = \frac{i\omega}{4\pi}\frac{\sqrt{\sin\theta}}{b_o^{5/2}\sqrt{\rho_o}}\big|[\nabla R, \vec{l}\,]\big|\,. \tag{7.66}$$

Clearly, we must consider $\psi_o^{(2)}(\theta,\phi) = 0$ in this case.

Remark.

Consider an unbounded homogeneous medium and assume for simplicity that $\vec{f} \equiv 0$. In this case we can introduce Helmholtz's potentials Θ and $\vec{\Psi}$ as follows

$$\vec{U} = \text{grad}\ \Theta\ , \quad \vec{U} = \text{rot}\ \vec{\Psi}\ .$$

By inserting these formulas into the elastodynamic equations (7.48) or (7.49), we can reduce them to the independent equations for Θ and $\vec{\Psi}$ as shown above. Then, in general, a solution of elastodynamic equations can be presented as the superposition

$$\vec{U} = \text{grad}\ \Theta + \text{rot}\ \vec{\Psi}\ .$$

If, however, there is an interface separating two different homogeneous media then, usually, both Θ and $\vec{\Psi}$ should be taken into consideration in order to satisfy boundary conditions on the interface. So, in such cases, problems for Θ and $\vec{\Psi}$ cannot be split.

7.8 The ray method in a medium with smooth interfaces

If an inhomogeneous medium contains smooth interfaces, some boundary conditions should be imposed on the interfaces. They involve, in general, conditions for the total displacement vector and the stress tensor. In order to satisfy the boundary conditions we should follow the rationale used in the theory of plane wave propagation in elastodynamics. But this time we have to take into account only the main terms, i.e. terms which contain the large parameter ω, if we work with the leading term of the ray series. Normally, on the interface we obtain two reflected and two transmitted elastic waves.

Let us consider a sequence of steps which should be taken to satisfy the boundary conditions within a zero-order ray approximation.

Let us use the following notations

$$\begin{aligned}
\vec{U}_{in} &= \vec{A}_{in} e^{i\omega\tau_{in}} && \text{for incident wave,}\\
\vec{U}_{r}^{(p)} &= \vec{P}_{r} e^{i\omega\tau_{r}^{(p)}} && \text{for reflected } P \text{ wave,}\\
\vec{U}_{r}^{(s)} &= \vec{S}_{r} e^{i\omega\tau_{r}^{(s)}} && \text{for reflected } S \text{ wave,}\\
\vec{U}_{tr}^{(p)} &= \vec{P}_{tr} e^{i\omega\tau_{tr}^{(p)}} && \text{for transmitted } P \text{ wave,}\\
\vec{U}_{tr}^{(s)} &= \vec{S}_{tr} e^{i\omega\tau_{tr}^{(s)}} && \text{for transmitted } S \text{ wave,}
\end{aligned}$$

by omitting everywhere the time-dependent multiplier $\exp\{-i\omega t\}$.

1. If we have only one incident wave in medium 1, we have to employ both reflected P and S waves. Hence, the total displacement vector in medium 1 reads
$$\vec{U}^{(1)} = \vec{U}_{in} + \vec{U}_{r}^{(p)} + \vec{U}_{r}^{(s)}\ .$$

In medium 2 we obtain

$$\vec{U}^{(2)} = \vec{U}_{tr}^{(p)} + \vec{U}_{tr}^{(s)} \,.$$

2. On the next step we have to gather the main terms of the stress tensor τ_{jk}. To clarify the situation, let us look at a derivative of the displacement vector. For example,

$$\begin{aligned} \text{div}\ \vec{U}^{(1)} &= \text{div}\ \left(\vec{U}_{in} + \vec{U}_r^{(p)} + \vec{U}_r^{(s)}\right) = \\ &= e^{i\omega\tau_{in}}[i\omega(\nabla\tau_{in}, \vec{A}_{in}) + \text{div}\ \vec{A}_{in}] + \\ &\quad + e^{i\omega\tau_r^{(p)}}\left[i\omega(\nabla\tau_r^{(p)}, \vec{P}_r) + div\vec{P}_r\right] + \\ &\quad + e^{i\omega\tau_r^{(s)}}\left[i\omega\big(\nabla\tau_r^{(s)}, \vec{S}_r\big) + \ \text{div}\ \vec{S}_r\right] . \end{aligned}$$

It is clear now, that the large parameter ω appears as a multiplier if and only if we differentiate the exponents, and amplitudes $\vec{A}, \vec{S}_r, \vec{P}_r$ are not differentiated in the main terms, i.e. they may be considered as constants (compare with plane waves in homogeneous media!).

Such terms like div $\vec{A}$, div $\vec{P}_r$, div $\vec{S}_r$ do not contain the large parameter ω and because of that they may be omitted in the boundary conditions, if we deal with the main term of the ray series. Thus, by introducing the wave vector $\vec{k}$ through the formula $\vec{k} = \omega\nabla\tau$, we can eventually deduce that the leading terms of the stress tensor will be precisely the same as in case of plane waves in a homogeneous medium. But unlike the latter case, they have to be calculated on a curved interface in an inhomogeneous medium and therefore they remain to be functions of a point on the interface.

3. By inserting the displacement vector and the stress tensor into the boundary conditions we impose the following requirements on the eikonals

$$\tau_{in}\big|_S = \tau_r^{(p)}\big|_S = \tau_r^{(s)}\big|_S = \tau_{tr}^{(p)}\big|_S = \tau_{tr}^{(s)}\big|_S$$

which give rise to Snell's law for the reflected and transmitted rays.

4. Now the exponents can be canceled from the boundary equations and we arrive at a linear system of algebraic equations for the amplitudes of the reflected and transmitted waves. Note that this system is precisely the same which we had for the plane waves in homogeneous media with flat interfaces! Hence, we can conclude now that the reflection and the transmission coefficients coincide with those known in the case of plane waves in homogeneous media. But now they depend upon coordinates of the point of incidence on the interface!

7.9 Second term of the ray series and the problem of validity of the ray theory

Let us derive the formula for the second term $\vec{u}_1$ of the ray series in the case of P-waves. To this end we have to consider the second and the third equations in (7.28).

We seek the second term in the following form

$$\vec{u}_1 = \varphi_1 a \nabla\tau + \vec{u}_1^{(o)}\,, \tag{7.67}$$

where $\vec{u}_1^{(o)}$ is supposed to be orthogonal to $\nabla\tau$. By inserting expression (7.67) into the second equation (7.28) and taking into account that $N_k(\nabla\tau) = 0$, $k = 1, 2, 3$, we obtain

$$N_k\big(\vec{u}_1^{(o)}\big) = -M_k(\vec{u}_o)\,. \tag{7.68}$$

In order to derive the transport equation for φ_1 we have to consider equation (7.28) for $\vec{u}_2$. Now, to make this equation solvable with respect to $\vec{u}_2$ we must impose the following condition of orthogonality of the right-hand side to the solution of homogeneous equation

$$\big(-\vec{M}(\vec{u}_1) - \vec{L}(\vec{u}_o)\,,\, \nabla\tau\big) = 0. \tag{7.69}$$

Let us develop equations (7.68) and (7.69) by employing expressions for operators N_k and M_k (see section 7.4). As the mixed term $\vec{u}_1^{(o)}$ is orthogonal to $\nabla\tau$ and $\Lambda = -(\lambda + \mu)/a^2$ in the case under consideration, we obtain the following expression for it

$$\vec{u}_1^{(o)} = \frac{a^2}{\lambda + \mu}\vec{M}(\vec{u}_o)\,. \tag{7.70}$$

Note that the right-hand side of equation (7.70) is orthogonal to $\nabla\tau$. Indeed, $\vec{u}_o = \varphi_o a \nabla\tau$ and the transport equation for φ_o was derived from the equation of orthogonality $(\vec{M}(\vec{u}_o), \nabla\tau) = 0$.

Consider further the scalar product $(\vec{M}(\vec{u}_1), \nabla\tau)$. Taking formula (7.67) into account we get

$$(\vec{M}(\vec{u}_1), \nabla\tau) = (\vec{M}(\varphi_1 a \nabla\tau), \nabla\tau) + (\vec{M}(\vec{u}_1^{(o)}), \nabla\tau)\,, \tag{7.71}$$

where the first term gives rise to the left-hand side of the transport equation for φ_1 (see section 7.6). The second term in (7.71) should be regarded as an inhomogeneous term for the transport equation.

By repeating the calculations in section 7.6, we get from the condition of orthogonality (7.69) the following equation for φ_1

$$\frac{1}{a}\big\{2(\nabla\tau, \nabla\varphi_1)a^2\rho + a^2\rho\varphi_1\Delta\tau + \varphi_1(\nabla\tau, \nabla a^2\rho)\big\} = -(\vec{M}(\vec{u}_1^{(o)}) + \vec{L}(\vec{u}_o), \nabla\tau). \tag{7.72}$$

Obviously, equation (7.72) is an inhomogeneous transport equation. By integrating this equation by means of the procedure described in section 3.4, we arrive at the final result

$$\varphi_1 = \frac{1}{\sqrt{a^2\rho}\sqrt{\frac{1}{a}J}}\left\{\psi_1(\alpha,\beta) - \frac{1}{2}\int_0^\tau \frac{a^2}{\sqrt{\rho}}\sqrt{\frac{1}{a}J}(\vec{M}(\vec{u}_1^{(o)}) + \vec{L}(\vec{u}_o), \nabla\tau)d\tau\right\}\ , \tag{7.73}$$

where $\psi_1(\alpha,\beta)$ is a constant of integration, or initial value of φ_1 at a point $\tau = 0$.

Apparently, similar calculations can be carried out for S waves. A corresponding expression for the second term will contain the mixed term as well. But in that case the mixed term is collinear with $\nabla\tau$.

In order to study the possible behavior of the second term with respect to the distance to a source let us explore the following particular case.

Consider a 3D homogeneous medium. Suppose we have initially a plane wave front with varying amplitude along it. Let it coincide with the z, y-plane and study its propagation along the x axis. A corresponding family of rays is formed by straight lines parallel to the x axis. Let the ray parameters γ_1, γ_2 be the coordinates on the y, z-plane, e.g. $\gamma_1 = y$ and $\gamma_2 = z$, then the family of rays is described by formulas

$$x = a\tau, \quad y = \gamma_1, \quad z = \gamma_2\ , \tag{7.74}$$

where τ is the eikonal. It follows from equation (7.74) that

$$J = \frac{1}{a}\left|\frac{D(x,y,z)}{D(\tau,\gamma_1,\gamma_2)}\right| = \frac{1}{a}a = 1\ .$$

According to equations (7.26) and (7.45) we get

$$\vec{u}_o = \frac{\psi_o(\gamma_1,\gamma_2)}{\sqrt{\rho a}}a\nabla\tau = \frac{\psi_o(y,z)}{\sqrt{\rho a}}\vec{i}\ . \tag{7.75}$$

Thus, $\vec{u}_o$ depends upon coordinates only because of function ψ_o which describes the initial value of the amplitude.

Consider further the mixed term $\vec{u}_1^{(o)}$ given by formula (7.70). For a homogeneous medium we deduce from equation (7.33)

$$\vec{M}(\vec{A}) = (\lambda+\mu)\big\{\nabla\tau \text{ div } \vec{A} + \nabla(\nabla\tau,\vec{A})\big\} + \mu\big[2(\nabla\tau,\nabla)\vec{A} + \Delta\tau\vec{A}\big]\ . \tag{7.76}$$

In the case under consideration $\tau = x/a$ which yields $\Delta\tau = 0$, $(\nabla\tau,\nabla) = (1/a)(\partial/\partial x)$.

Then, taking into account that $\vec{u}_o$ does not depend on x and by inserting $\vec{u}_o$ into equation (7.76) we obtain $\vec{M}(\vec{u}_o) = [(\lambda+\mu)/(a\sqrt{\rho a})]\nabla\psi_o$ and therefore

$$\vec{u}_1^{(o)} = \sqrt{\frac{a}{\rho}}\nabla\psi_o = \sqrt{\frac{a}{\rho}}\left(\frac{\partial\psi_o}{\partial y}\vec{j} + \frac{\partial\psi_o}{\partial z}\vec{k}\right)\ . \tag{7.77}$$

Note that $\nabla\psi_o$ is orthogonal to the direction of propagation, i.e. to the x-axis.

Thus, we come to the conclusion that if distribution of the amplitude along the initial wave front is not uniform, i.e. ψ_o does depend upon the ray parameters, the mixed term $\vec{u}_1^{(o)}$ appears even in a homogeneous medium. In our particular case it remains to be constant along each ray.

Consider now the integral in formula (7.73). By inserting expression (7.77) into equation (7.76) we obtain

$$\vec{M}(\vec{u}_1^{(o)}) = (\lambda + \mu)\nabla\tau\sqrt{\frac{a}{\rho}}\ \text{div}\ \nabla\psi_o\,,$$

and therefore

$$(\vec{M}(\vec{u}_1^{(o)}), \nabla\tau) = \frac{\lambda+\mu}{a\sqrt{\rho a}}\Delta\psi_o = \frac{\lambda+\mu}{a\sqrt{\rho a}}\left(\frac{\partial^2\psi_o}{\partial y^2} + \frac{\partial\psi_o}{\partial z^2}\right)\,. \tag{7.78}$$

For a homogeneous medium the expression for operator $\vec{L}(\vec{A})$ takes the form (see section 7.4)

$$\vec{L}(\vec{A}) = (\lambda + \mu)\nabla\ \text{div}\ \vec{A} + \mu\Delta\vec{A}\,. \tag{7.79}$$

By inserting expression (7.75) for $\vec{u}_o$ into equation (7.79) and taking into account that $\vec{u}_o$ does not depend upon x we obtain

$$\big(\vec{L}(\vec{u}_o), \nabla\tau\big) = \frac{\mu}{a\sqrt{\rho a}}\Delta\psi_o\,. \tag{7.80}$$

Now we are able to derive the final formula for the first item of the second term (7.67) of the ray series. Indeed, by inserting formulas (7.78), (7.80) into equation (7.73) we obtain

$$\varphi_1 = \frac{1}{\sqrt{a\rho}}\left\{\psi_1(\gamma_1,\gamma_2) - \frac{1}{2}a^2\Delta\psi_o\tau\right\} \tag{7.81}$$

where $\Delta\psi_o = \partial^2\psi_o/\partial y^2 + \partial^2\psi_o/\partial z^2$ because ψ_o does not depend upon x - see equation (7.75).

Based on the final formula (7.81) we arrive at the following conclusion: if the distribution of the amplitude on the initial wave front is not uniform, and $\Delta\psi_o$ does not equal to zero identically, the magnitude of the second term increases proportionally to the distance from the initial wave front. According to the criterion given in Chapter 1 we obtain a limitation to the validity of the ray method formulas with respect to the distance, if the frequency is fixed, or with respect to frequency, if the distance is fixed. But is this the only instance when the ray method fails with an increasing distance to the source?

The following interpretation to the result described above can be given. Being within the frames of the zero-order ray approximation, we can construct a beam of the wave field which propagates along a straight line without spreading. Indeed, in the case under consideration we have a family of rays parallel to the x-axis. By varying the amplitude along the initial wave front, this ray field can be bounded in lateral directions. To this end $\psi_o(\gamma_1, \gamma_2)$ should be identically equal to zero out of some domain on the initial wave front.

Due to the fact that the energy propagates along the ray tubes and it has no transversal diffusion, all energy of the initial wave field will remain in that beam. And this beam does not spread in lateral directions at all because the geometrical spreading is constant and independent of x - see equation (7.75)!

But such type of wave propagation phenomenon seems to be unrealistic and incorrect from the physical point of view. Evidence of that exists in the ray theory too but only if the second term of the ray series is taken into account! We can conclude that the ray method fails in this particular case because it does not properly describe the transversal diffusion of energy.

Let us dwell on another phenomenon related to the second term of the ray series. The second term contains the so-called mixed term. For example: in case of P-wave $\vec{u}_1^{(o)}$ being orthogonal to $\nabla\tau$ - see equation (7.67).

This mixed term is responsible for a de-polarization phenomenon, which consists of the following. If, for instance, we take into account two terms of the ray series for a P wave, its polarization will not be exactly parallel to the direction of propagation. The same happens to the S waves.

Note that such type of de-polarization phenomenon can be obtained in a slightly anisotropic but homogeneous medium. Therefore, by observing the depolarization phenomenon in real geophysical explorations we have to decide whether it is caused by anisotropy or it can be explained in the frames of inhomogeneous but isotropic models.

In order to investigate this problem we need an algorithm for the computation of the second term of the ray series. Thus the problem of computation of the second term of the ray series is becoming rather an actual and important one.

An algorithm for computation of the second term based on ideas of the paraxial ray theory has been developed in the paper by Kirpichnikova, Popov and Pšenčík (1997). For more results of its application to the problem of validity of the ray theory see Popov and Camerlynck (1996) and Popov and Oliveira (1997).

8

The Gaussian Beam method in elastodynamics. The frequency domain

8.1 Construction of a Gaussian beam

Consider an arbitrary ray in an inhomogeneous elastic medium, which we denote by $\vec{r}_o = \vec{r}_o(s)$, where s is the arc length along the ray.

Let us formulate the following problem.

Consider a vicinity of this ray $\vec{r}_o(s)$ and let us try to construct an approximate solution of elastodynamic equations which possesses the following properties

i) the solution should be concentrated in a vicinity of the central ray $\vec{r}_o(s)$ and decline quickly with an increasing distance to the central ray,

ii) the solution should not have any singularity along the ray.

From a first glance, it is not evident at all that such type of approximate solutions could exist, but they were discovered in the 1960s by specialists in the theory of gas lasers. Those solutions were implemented later for elastodynamic equations by Kirpichnikova (1971) (for more details see the Introduction).

We shall consider only the main term of such approximate solutions because it provides a significant simplification in their constructions.

P - Gaussian beams.

We know now that we can satisfy the elastodynamic equations (approximately as $\omega \to \infty$) by the following vector - function

$$\vec{U} = \vec{A}e^{i\omega\tau} = \frac{\tilde{\varphi}_o}{\sqrt{a^2\rho}} e^{i\omega\tau} a\nabla\tau \tag{8.1}$$

if the eikonal τ satisfies the eikonal equation

$$(\nabla\tau)^2 = \frac{1}{a^2} \tag{8.2}$$

and $\tilde{\varphi}_o$ is a solution of the transport equation

$$2(\nabla\tau, \nabla\tilde{\varphi}_o) + \tilde{\varphi}_o\Delta\tau = 0\,. \tag{8.3}$$

According to our first statement (i) we may consider equations (8.2) and (8.3) in a vicinity of a given central ray $\vec{r}_o(s)$. To this end we use the ray centered coordinates s, q_1, q_2 and seek the eikonal τ in the form

$$\tau(s, q_1, q_2) = \tau_o(s) + \frac{1}{2}\sum_{j,k=1}^{2}\Gamma_{jk}(s)q_j q_k + \cdots\,. \tag{8.4}$$

Accordingly, for $\tilde{\varphi}_o$ we consider

$$\tilde{\varphi}_o(s, q_1, q_2) = \tilde{\varphi}_{00}(s) + \cdots\,. \tag{8.5}$$

In order to construct the main term of a Gaussian beam we may restrict ourselves to two terms in equation (8.4) and the first term in equation (8.5).

Now we can apply the results obtained in the paraxial ray theory.

Suppose we have two solutions $X^{(1)}(s)$ and $X^{(2)}(s)$ of the equations in variations

$$\begin{aligned} \frac{dq_j}{ds} &= \frac{\partial H_2}{\partial p_j}\,, \quad \frac{dp_j}{ds} = -\frac{\partial H_2}{\partial q_j}\,, \quad j = 1, 2\,, \\ H &= -\frac{h}{a}\sqrt{1 - a^2(p_1^2 + p_2^2)} = H_o(s) + H_2 + \cdots\,. \end{aligned}$$

By means of these solutions we can construct two matrices 2 x 2

$$\mathbf{Q} = \begin{pmatrix} q_1^{(1)} & q_1^{(2)} \\ q_2^{(1)} & q_2^{(2)} \end{pmatrix}\,, \quad \mathbf{P} = \begin{pmatrix} p_1^{(1)} & p_1^{(2)} \\ p_2^{(1)} & p_2^{(2)} \end{pmatrix}\,. \tag{8.6}$$

If the $det\,\mathbf{Q} \neq 0$, i.e. the matrix $\mathbf{Q}$ is not singular for all possible values of s, we can construct a matrix $\mathbf{\Gamma} = \mathbf{P}\mathbf{Q}^{-1}$ which will satisfy Ricatti's equation and hence the eikonal (8.4) will be an approximate solution of the eikonal equation (8.2).

Accordingly, we obtain the following formula for the main term of expansion (8.5) for the "amplitude" $\tilde{\varphi}_o$,

$$\tilde{\varphi}_{oo}(s) = \frac{\text{const}}{\sqrt{\dfrac{1}{a_o}\det\mathbf{Q}}}\,, \tag{8.7}$$

and we finally arrive at the following approximate solution of the elastodynamic equations in a vicinity of the central ray $\vec{r}_o(s)$

$$\vec{U} = \frac{\text{const}}{\sqrt{\frac{1}{a_o}\det \mathbf{Q}}} \frac{\vec{t}}{\sqrt{a_o^2\rho_o}} \exp\left[i\omega\left[\tau_o + \frac{1}{2}\sum_{j,k=1}^{2} \Gamma_{jk} q_j q_k\right]\right] , \tag{8.8}$$

where $\vec{t}(s) = d\vec{r}_o(s)/ds$ is the unit vector tangent to the central ray and by a_o and ρ_o we denote the velocity of P-waves a and the density ρ calculated on the central ray, respectively.

S - Gaussian beams.

In this case we have the following formula for S - waves in the zero-order approximation of the ray theory

$$\vec{U}_j = \frac{\tilde{B}_j}{\sqrt{b^2\rho}} \vec{e}_j e^{i\omega\tau} , \quad j = 1, 2 . \tag{8.9}$$

where $\vec{e}_j, j = 1, 2$, are unit vectors of the ray centered coordinates and τ and $\tilde{B}_j$ should satisfy the eikonal

$$(\nabla\tau)^2 = \frac{1}{b^2}$$

and the transport equations

$$2(\nabla\tau, \nabla\tilde{B}_j) + \tilde{B}_j\Delta\tau = 0 ,$$

respectively. Then by simply repeating all calculations made in the case of P - waves, we obtain two mutually orthogonal approximate solutions of the elastodynamic equations

$$\vec{U}_j = \frac{\text{const}}{\sqrt{\frac{1}{b_o}\det \mathbf{Q}}} \frac{\vec{e}_j}{\sqrt{b_o^2\rho_o}} \exp\left[i\omega\left[\tau_o(s) + \frac{1}{2}\sum_{j,k=1}^{2} \Gamma_{jk} q_j q_k\right]\right] , \quad j = 1, 2 . \tag{8.10}$$

Suppose we can now choose such solutions of the equations in variations that the following properties of the matrices (8.6) and $\mathbf{\Gamma}$ will hold true:

1) $\det \mathbf{Q} \neq 0$ for arbitrary s,
2) the matrices $\mathbf{\Gamma}$ are symmetrical, i.e. $\mathbf{\Gamma}^T = \mathbf{\Gamma}$, complex valued and their imaginary parts are positive-defined.

Then both approximate solutions (8.8) and (8.10) will have no singularities for arbitrary s and will be concentrated in a vicinity of the central ray $\vec{r}_o(s)$. The latter result comes out from the equality

$$\left| \exp\left[i\omega\left[\tau_o(s) + \frac{1}{2}\sum_{j,k=1}^{2} \Gamma_{jk} q_j q_k\right]\right]\right| = \exp\left[-\frac{\omega}{2}\sum_{j,k=1}^{2} \text{Im}\, \Gamma_{jk} q_j q_k\right] . \tag{8.11}$$

Indeed, it follows from equation (8.11) that the modulus of the wave field decreases exponentially with the increasing coordinates q_1, q_2 and therefore with an increasing distance to the central ray.

8.2 How to choose suitable solutions for the equations in variations

Consider first some auxiliary formulas. Let us denote by $X^{(j)}$ two complex valued solutions of the equations in variations

$$X^{(j)} = \begin{pmatrix} q_1^{(j)} \\ q_2^{(j)} \\ p_1^{(j)} \\ p_2^{(j)} \end{pmatrix} \equiv \begin{pmatrix} q^{(j)} \\ p^{(j)} \end{pmatrix} , \quad j = 1, 2 ,$$

where $q^{(j)}$ and $p^{(j)}$ now mean 2D column vectors

$$q^{(j)} = \begin{pmatrix} q_1^{(j)} \\ q_2^{(j)} \end{pmatrix} \text{ and } p^{(j)} = \begin{pmatrix} p_1^{(j)} \\ p_2^{(j)} \end{pmatrix} .$$

By using these new notations we can present the $\mathbf{J}$ - scalar product in the form

$$\left(\mathbf{J}X^{(1)}, X^{(2)}\right) = \left(p^{(1)}, q^{(2)}\right) - \left(q^{(1)}, p^{(2)}\right) ,$$

where as usual

$$\left(p^{(1)}, q^{(2)}\right) = p_1^{(1)} q_1^{(2)} + p_2^{(1)} q_2^{(2)} .$$

Now we establish the first auxiliary formula for matrices $\mathbf{Q}$ and $\mathbf{P}$

$$\mathbf{P}^T\mathbf{Q} - \mathbf{Q}^T\mathbf{P} = \begin{pmatrix} (\mathbf{J}X^{(1)}, X^{(1)}) & (\mathbf{J}X^{(1)}, X^{(2)}) \\ (\mathbf{J}X^{(2)}, X^{(1)}) & (\mathbf{J}X^{(2)}, X^{(2)}) \end{pmatrix} . \tag{8.12}$$

In fact, we have

$$\mathbf{P}^T\mathbf{Q} - \mathbf{Q}^T\mathbf{P} = \begin{pmatrix} (p^{(1)}, q^{(1)}) - (q^{(1)}, p^{(1)}) & (p^{(1)}, q^{(2)}) - (q^{(1)}, p^{(2)}) \\ (p^{(2)}, q^{(1)}) - (q^{(2)}, p^{(1)}) & (p^{(2)}, q^{(2)}) - (q^{(2)}, p^{(2)}) \end{pmatrix}$$

which coincides with the right-hand side of equation (8.12).

Further, let us denote by $\mathbf{P}^+$ the Hermitian conjugate matrix, i.e. $(\mathbf{P}^+)_{jk} = \bar{\mathbf{P}}_{kj}$. Then we get the following second auxiliary formula

$$\mathbf{Q}^+\mathbf{P} - \mathbf{P}^+\mathbf{Q} = \begin{pmatrix} (\mathbf{J}X^{(1)}, \bar{X}^{(1)}) & (\mathbf{J}X^{(2)}, \bar{X}^{(1)}) \\ (\mathbf{J}X^{(1)}, \bar{X}^{(2)}) & (\mathbf{J}X^{(2)}, \bar{X}^{(2)}) \end{pmatrix} \tag{8.13}$$

where $\bar{X}^{(j)}$ is a complex conjugate vector with respect to the vector - column $X^{(j)}$.

Remark 1. Evidently, if $X^{(j)}$ is a solution of the equations in variations, then $\bar{X}^{(j)}$ also satisfies these equations because the equations do not contain complex coefficients.

Remark 2. Due to $(d/ds)(\mathbf{J}X^{(1)}, X^{(2)}) \equiv 0$ for arbitrary solutions $X^{(1)}$ and $X^{(2)}$ of the equations in variations both $\mathbf{P}^T\mathbf{Q} - \mathbf{Q}^T\mathbf{P}$ and $\mathbf{Q}^+\mathbf{P} - \mathbf{P}^+\mathbf{Q}$ do not depend upon s.

Then the third auxiliary formula reads

$$\text{Re}\ (\mathbf{J}X^{(j)}, \bar{X}^{(j)}) = 0 \quad \text{for}\ j = 1, 2\,. \tag{8.14}$$

Proof : we have

$$\begin{aligned} &\big(\mathbf{J}X^{(j)}, \bar{X}^{(j)}\big) = \big(\bar{X}^{(j)}, \mathbf{J}X^{(j)}\big) = \big(\mathbf{J}^T\bar{X}^{(j)}, X^{(j)}\big) = \\ &= -\big(\mathbf{J}\bar{X}^{(j)}, X^{(j)}\big) = -\overline{\big(\mathbf{J}X^{(j)}, \bar{X}^{(j)}\big)} \end{aligned}$$

which means that the real part of this $\mathbf{J}$ - scalar product is equal to zero identically with respect to s.

Assume now that $X^{(j)}, j = 1, 2$, are chosen in such a way that

$$\begin{aligned} \big(\mathbf{J}X^{(1)}, X^{(2)}\big) &= 0 \\ \big(\mathbf{J}X^{(1)}, \bar{X}^{(2)}\big) &= 0 \\ \big(\mathbf{J}X^{(j)}, \bar{X}^{(j)}\big) &= i\gamma_j^2\,, \quad \text{Im}\ \gamma_j = 0\,, \quad j = 1, 2\,. \end{aligned} \tag{8.15}$$

It follows now from equation (8.12) and the first one from equations (8.15) that

$$\mathbf{P}^T\mathbf{Q} - \mathbf{Q}^T\mathbf{P} = 0 \tag{8.16}$$

due to $(\mathbf{J}X^{(j)}, X^{(j)}) = 0$ for arbitrary j.

From equations (8.15) and (8.13), (8.14) we get

$$\mathbf{Q}^+\mathbf{P} - \mathbf{P}^+\mathbf{Q} = i\begin{pmatrix} \gamma_1^2 & 0 \\ 0 & \gamma_2^2 \end{pmatrix}\,. \tag{8.17}$$

Now we are able to prove the following statement: if $\det\mathbf{Q} \neq 0$ and $\mathbf{\Gamma} = \mathbf{P}\mathbf{Q}^{-1}$ is therefore well defined, then

i) $\mathbf{\Gamma}$ is symmetrical, i.e. $\mathbf{\Gamma}^T = \mathbf{\Gamma}$,

ii) the imaginary part of $\mathbf{\Gamma}$, i.e. Im $\mathbf{\Gamma} = \frac{1}{2i}(\mathbf{\Gamma} - \mathbf{\Gamma}^+)$ is positively defined.

Proof: consider the difference $\mathbf{\Gamma} - \mathbf{\Gamma}^T$, which can be developed as follows

$$\begin{aligned} &\mathbf{\Gamma} - \mathbf{\Gamma}^T = \mathbf{P}\mathbf{Q}^{-1} - (\mathbf{P}\mathbf{Q}^{-1})^T = \mathbf{P}\mathbf{Q}^{-1} - (\mathbf{Q}^T)^{-1}\mathbf{P}^T = \\ &= (\mathbf{Q}^T)^{-1}\{\mathbf{Q}^T\mathbf{P} - \mathbf{P}^T\mathbf{Q}\}\mathbf{Q}^{-1} = 0 \end{aligned}$$

due to equation (8.16) holding true. Hence, $\mathbf{\Gamma} = \mathbf{\Gamma}^T$.

Consider then $\mathbf{\Gamma} - \mathbf{\Gamma}^+$, which can be developed as follows

$$\begin{aligned}&\mathbf{\Gamma} - \mathbf{\Gamma}^+ = \mathbf{P}\mathbf{Q}^{-1} - (\mathbf{P}\mathbf{Q}^{-1})^+ = \mathbf{P}\mathbf{Q}^{-1} - (\mathbf{Q}^+)^{-1}\mathbf{P}^+ = \\ &= (\mathbf{Q}^+)^{-1}\{\mathbf{Q}^+\mathbf{P} - \mathbf{P}^+\mathbf{Q}\}\mathbf{Q}^{-1}\,.\end{aligned}$$

Taking now into account equation (8.17) we obtain

$$\text{Im}\,\mathbf{\Gamma} = \frac{1}{2i}(\mathbf{\Gamma} - \mathbf{\Gamma}^+) = (\mathbf{Q}^+)^{-1}\begin{pmatrix} \frac{1}{2}\gamma_1^2 & 0 \\ 0 & \frac{1}{2}\gamma_2^2 \end{pmatrix}\mathbf{Q}^{-1} \tag{8.18}$$

which implies that $\text{Im}\,\mathbf{\Gamma}$ is a positively defined matrix.

So all that is left to be proven is that $\mathbf{Q}$ is not a singular matrix, i.e. $\det\mathbf{Q} \neq 0$ for each s.

Here is the proof.

Suppose that at some point s_* we have $\det\mathbf{Q} = 0$. This means that the columns of the matrix $\mathbf{Q}$ are linearly dependent, i.e. there are some constants $a_1 \neq 0$, and $a_2 \neq 0$ in which the following equations hold true

$$\begin{aligned} a_1 q_1^{(1)}(s_*) + a_2 q_1^{(2)}(s_*) &= 0\,, \\ a_1 q_2^{(1)}(s_*) + a_2 q_2^{(2)}(s_*) &= 0\,. \end{aligned} \tag{8.19}$$

We can rewrite the system (8.19) in the vector form as follows

$$a_1 q^{(1)}(s_*) + a_2 q^{(2)}(s_*) = 0\,. \tag{8.20}$$

Let us come back to equations (8.15). We can present them in the form

$$\begin{aligned} \big(p^{(1)}, \bar{q}^{(2)}\big) - \big(q^{(1)}, \bar{p}^{(2)}\big) = 0 \quad &\leftrightarrow \quad \big(JX^{(1)}, \bar{X}^{(2)}\big) = 0 \\ \big(p^{(2)}, \bar{q}^{(1)}\big) - \big(q^{(2)}, \bar{p}^{(1)}\big) = 0 \quad &\leftrightarrow \quad \big(JX^{(2)}, \bar{X}^{(1)}\big) = 0 \\ \big(p^{(1)}, \bar{q}^{(1)}\big) - \big(q^{(1)}, \bar{p}^{(1)}\big) = i\gamma_1^2 \quad &\leftrightarrow \quad \big(JX^{(1)}, \bar{X}^{(1)}\big) = i\gamma_1^2 \\ \big(p^{(2)}, \bar{q}^{(2)}\big) - \big(q^{(2)}, \bar{p}^{(2)}\big) = i\gamma_2^2 \quad &\leftrightarrow \quad \big(JX^{(2)}, \bar{X}^{(2)}\big) = i\gamma_2^2 \end{aligned}$$

Let us multiply all equations by $a_1\bar{a}_2$, $\bar{a}_1 a_2$, $a_1\bar{a}_1$ and $a_2\bar{a}_2$, respectively, and then summarize all of them. That results in the following formula

$$\begin{aligned} &(a_1 p^{(1)}, \overline{a_2 q^{(2)}} + \overline{a_1 q^{(1)}}) + (a_2 p^{(2)}, \overline{a_1 q^{(1)}} + \overline{a_2 q^{(2)}}) - \\ &\quad - (a_1 q^{(1)} + a_2 q^{(2)}, \overline{a_2 p^{(2)}}) - (a_2 q^{(2)} + a_1 q^{(1)}, \overline{a_1 p^{(1)}}) = \\ &\qquad = i\gamma_1^2 |a_1|^2 + i\gamma_2^2 |a_2|^2\,. \end{aligned}$$

By applying equation (8.20) to the latter formula and bearing in mind that $\overline{a_1 q^{(1)}(s_*) + a_2 q^{(2)}(s_*)} = 0$, we obtain

$$0 = i\big[\gamma_1^2 |a_1|^2 + \gamma_2^2 |a_2|^2\big]$$

which yields that $a_1 = a_2 = 0$. This contradiction proves that $\det\mathbf{Q} \neq 0$ for the arbitrary value of the arc length s along the central ray.

Conclusion. Thus, if we choose two complex valued solutions $X^{(1)}(s)$ and $X^{(2)}(s)$ of the equations in variations in accordance with the formulas (8.15) and insert them into expressions (8.8) and (8.10) for Gaussian beams, we shall obtain approximate solutions of the elastodynamic equations which possess the properties listed in the beginning of this chapter: they are concentrated in a vicinity of the central ray $\vec{r}_o(s)$ and have no singularities all along.

Note that the parameters γ_1, and γ_2 in equations (8.15) remain arbitrary parameters of a Gaussian beam. They influence on the width of the Gaussian beam and we shall discuss their role later in more details.

8.3 Example 1: a 3D homogeneous medium

In this case, vectors $\vec{e}_1(s)$ and $\vec{e}_2(s)$ do not depend upon s due to $\kappa_j = (1/C)(\partial C/\partial q_j)\big|_{q_1=q_2=0} = 0$ where C is the velocity of waves (for P - waves $C = a$ and $C = b$ for S - waves).

The equations in variations read ($C_0 = C$)

$$\frac{dq_j}{ds} = C_o p_j \, , \quad \frac{dp_j}{ds} = 0 \, , \quad j = 1, 2 \, .$$

The general solution can be written as follows

$$\begin{aligned} q_j(s) &= C_o p_j(s_o)(s - s_o) + q_j(s_o) \quad , \\ p_j(s) &= p_j(s_o) \, , \qquad\qquad j = 1, 2 \, . \end{aligned}$$

Let us set two complex solutions $X^{(1)}(s)$ and $X^{(2)}(s)$ by the following initial conditions at the point $s = s_o$

$$X^{(1)}(s_o) = \begin{pmatrix} 1 \\ 0 \\ \frac{i}{2}\gamma_1^2 \\ 0 \end{pmatrix} \, , \quad X^{(2)}(s_o) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{i}{2}\gamma_2^2 \end{pmatrix} \, .$$

It follows from the above that for $s = s_o$

$$\begin{aligned} \left(JX^{(1)}, X^{(2)}\right) &= 0 \, , \quad \left(JX^{(1)}, \bar{X}^{(2)}\right) = 0 \, , \\ \left(JX^{(j)}, \bar{X}^{(j)}\right) &= i\gamma_j^2 \, , \quad j = 1, 2 \, , \end{aligned}$$

and, obviously, they hold true for all s.

Further, we get for an arbitrary s

$$X^{(1)}_{(s)} = \begin{pmatrix} C_o \frac{i}{2}\gamma_1^2(s - s_o) + 1 \\ 0 \\ \frac{i}{2}\gamma_1^2 \\ 0 \end{pmatrix} \, , \quad X^{(2)}_{(s)} = \begin{pmatrix} 0 \\ C_o \frac{i}{2}\gamma_2^2(s - s_o) + 1 \\ 0 \\ \frac{i}{2}\gamma_2^2 \end{pmatrix}$$

and therefore

$$\mathbf{Q} = \begin{pmatrix} C_o \frac{i}{2}\gamma_1^2(s-s_o)+1 & 0 \\ 0 & C_o \frac{i}{2}\gamma_2^2(s-s_o)+1 \end{pmatrix},$$

$$\mathbf{P} = \begin{pmatrix} \frac{i}{2}\gamma_1^2 & 0 \\ 0 & \frac{i}{2}\gamma_2^2 \end{pmatrix}; \quad \det \mathbf{Q} = \left(C_o \frac{i}{2}\gamma_1^2(s-s_o)+1\right)\cdot\left(C_o \frac{i}{2}\gamma_2^2(s-s_o)+1\right).$$

The formula for $\boldsymbol{\Gamma}$ reads

$$\boldsymbol{\Gamma} = \mathbf{P}\mathbf{Q}^{-1} = \begin{pmatrix} \dfrac{\frac{i}{2}\gamma_1^2}{C_o\frac{i}{2}\gamma_1^2(s-s_o)+1} & 0 \\ 0 & \dfrac{\frac{i}{2}\gamma_2^2}{C_o\frac{i}{2}\gamma_2^2(s-s_o)+1} \end{pmatrix}$$

and for the exponent of a Gaussian beam we get

$$\exp\left\{ i\omega \left[\frac{s-s_o}{C_o} + \frac{1}{2}\Gamma_{11}(s)q_1^2 + \frac{1}{2}\Gamma_{22}(s)q_2^2 \right]\right\}$$

where

$$\Gamma_{jj}(s) = \frac{\frac{i}{2}\gamma_j^2}{C_o\frac{i}{2}\gamma_j^2(s-s_o)+1}, \quad j=1,2.$$

After some algebra we get

$$\operatorname{Re}\Gamma_{jj} = \frac{C_o\dfrac{\gamma_j^4}{4}(s-s_o)}{1+\dfrac{C_o^2\gamma_j^4}{4}(s-s_o)^2}, \quad \operatorname{Im}\Gamma_{jj} = \frac{\dfrac{1}{2}\gamma_j^2}{1+\dfrac{C_o^2\gamma_j^4}{4}(s-s_o)^2}.$$

Width of a Gaussian beam.

There is an important characteristic of a Gaussian beam which can be introduced by analogy with the Gaussian functions. Indeed, consider the function $g_j \equiv \exp\left\{-(\omega/2)\mathrm{Im}\,\Gamma_{jj}q_j^2\right\}$. If we present g_j in the form $g_j = \exp\left\{-q_j^2/\Lambda_j^2\right\}$, then Λ_j is called usually the half width of the Gaussian function g_j. In our case we get two half widths Λ_j, $j=1,2$, with respect to the ray centered coordinates q_1 and q_2, correspondingly,

$$\Lambda_j = \frac{1}{\sqrt{\dfrac{\omega}{2}\mathrm{Im}\,\Gamma_{jj}}} = \sqrt{\frac{1+\dfrac{C_o^2\gamma_j^4}{4}(s-s_o)^2}{\dfrac{\omega}{4}\gamma_j^2}}.$$

It follows from the latter formula that

i) Λ_j depends upon parameter γ_j and achieves the minimum at the point $s = s_o$,

ii) as $s \to \infty$ the half-width tends to infinity, i.e. $\Lambda_j \sim \dfrac{C_o \gamma_j s}{\sqrt{\omega}}$.

Thus, in our particular case a Gaussian beam has two independent widths $2\Lambda_j$ in the q_j direction, $j = 1, 2$. By means of the parameters s_o, γ_1 and γ_2 we can regulate the widths and position of their minimum. The width of a Gaussian beam increases monotonously and rather fast with the increase in distance $s - s_o$ along the central ray. Evidently, in inhomogeneous media the behavior of the width may be more complicated, but in any case a Gaussian beam will be narrower for a higher frequency ω

Consider, for example, curves on a q_j, s- plane defined by the condition $q_j^2/\Lambda_j^2 = K_j =$ const. In the case under consideration this yields a family of hyperbolae dependent on the constant K_j

$$\frac{q_j^2}{a^2} - \frac{(s - s_o)^2}{b^2} = 1\,, \quad a^2 = \frac{4K}{\omega \gamma_j^2}\,,\; b^2 = \frac{4}{C_o^2 \gamma_j^4}\,.$$

If we consider $K = 1$ we can conventionally say that the Gaussian beam is concentrated in the vicinity of the central ray bounded by two branches of the corresponding hyperbola.

By analogy with the ray method we may say that the term $\frac{1}{2}\sum_{i,j=1}^{2} \mathrm{Re}\,\mathbf{\Gamma}_{ij} q_i q_j$ in formulas (8.8) and (8.10) for the Gaussian beams describes the wavefronts of the beams.

Note that $\mathrm{Re}\,\Gamma_{jj} = 0$ for $s = s_o$ and therefore the wavefront of the Gaussian beam becomes flat at this point.

8.4 Reduction to a 2D

In a 2D case, unit vector $\vec{e}(s)$ of the ray centered coordinates is parallel to the normal of the central ray and it should be considered as known, if the central ray is known. The equations in variations have the form

$$\frac{d}{ds}q = C_o p\,,\; \frac{d}{ds}p = -\frac{1}{C_o^2}\left(\left.\frac{\partial^2 C}{\partial q^2}\right|_{q=0}\right) q\,, \tag{8.21}$$

where C is the velocity of the waves under consideration. We denote by $X^{(j)}(s)$, $j = 1, 2$, two solutions of the equations in variations

$$X^{(j)}(s) = \begin{pmatrix} q^{(j)}(s) \\ p^{(j)}(s) \end{pmatrix}\,.$$

Let us set $X^{(j)}$ by the following initial conditions

$$X^{(1)}(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad X^{(2)}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}\,,$$

then they are linearly independent for all s. Indeed, we can check that

$$\det \begin{pmatrix} q^{(1)}(s) & q^{(2)}(s) \\ p^{(1)}(s) & p^{(2)}(s) \end{pmatrix} = -\big(\mathbf{J}X^{(1)}, X^{(2)}\big) \,,$$

where this time $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. But the right-hand side of the latter equation does not depend on s and therefore $\big(\mathbf{J}X^{(1)}(s), X^{(2)}(s)\big) = \big(\mathbf{J}X^{(1)}(0), X^{(2)}(0)\big) = -1$.

Now let us introduce the following complex valued solution of the equations in variations

$$Q(s) = Z_1 q^{(1)}(s) + iZ_2 q^{(2)}(s) \,,\ P(s) = Z_1 p^{(1)}(s) + iZ_2 p^{(2)}(s) \tag{8.22}$$

where Z_1 and Z_2 are arbitrary positive constants. Then we get

$$\Gamma(s) = P(s)Q^{-1}(s) = \frac{Z_1 p^{(1)}(s) + iZ_2 p^{(2)}(s)}{Z_1 q^{(1)}(s) + iZ_2 q^{(2)}(s)} \,.$$

It is easy to see that

$$\text{Im}\ \Gamma(s) = \frac{-2iZ_1 Z_2 \big(\mathbf{J}X^{(1)}, X^{(2)}\big)}{2i|Q|^2} = \frac{Z_1 Z_2}{|Q|^2} > 0 \,.$$

Note that in the case under consideration $Q(s) \neq 0$ for all values of s. In fact, if $Q(s) = 0$ then both $q^{(1)}(s)$, $q^{(2)}(s)$ are equal to zero. But this means that $(\mathbf{J}X^{(1)}(s), X^{(2)}(s)) = 0$ which is impossible!

Hence, we obtain the following formulas for a P - Gaussian beam

$$\vec{U}^{(p)} = \frac{\text{const}}{\sqrt{a_o(s)\rho_o(s)}} \frac{\vec{t}(s)}{\sqrt{Q_a(s)}} \exp\left\{ i\omega \left[\tau_o^{(a)}(s) + \frac{1}{2}\Gamma_a(s) q^2 \right] \right\}$$

and for a S - Gaussian beam

$$\vec{U}^{(s)} = \frac{\text{const}}{\sqrt{b_o(s)\rho_o(s)}} \frac{\vec{e}(s)}{\sqrt{Q_b(s)}} \exp\left\{ i\omega \left[\tau_o^{(b)}(s) + \frac{1}{2}\Gamma_b(s) q^2 \right] \right\}$$

where indices a and b indicate the velocities of P and S waves, respectively.

Thus, to construct Gaussian beams in a 2D case we have to extend two real linearly independent solutions $X^{(j)}(s)$, $j = 1, 2$, of the equations in variations (8.21) along the central ray. Then a desirable complex valued solution can be obtained by formulas (8.22), where parameters Z_1 and Z_2 (more precisely only their ratio!) remain to be arbitrary (compare with γ_1, γ_2 in equations (8.15)).

8.5 Complex point source and a Gaussian beam

Consider a homogeneous medium and the Green's function problem for the reduced wave equation (in 3D)

$$\left(\Delta + \frac{\omega^2}{C^2} \right) G = -\delta(M - M_o) \,.$$

For G we have the following expression

$$G = \frac{1}{4\pi R} e^{i\frac{\omega}{C}R}\ ,\ R = \sqrt{(x-x_o)^2 + (y-y_o)^2 + (z-z_o)^2}\ .$$

Now let us "plunge" the source with coordinates x_o, y_o, z_o into the complex space, assuming for simplicity that $y_o = z_o = 0$ and $x_o = i\alpha^2$, $(i)^2 = -1$, $\mathrm{Im}\alpha = 0$.

Suppose further that $|x - x_o|^2 \gg y^2 + z^2$, then we can expand R as follows

$$\begin{aligned} R &= \sqrt{(x-x_o)^2\left(1 + \frac{y^2+z^2}{(x-x_o)^2}\right)} = (x-x_o)\left(1 + \frac{1}{2}\frac{y^2+z^2}{(x-x_o)^2} + \cdots\right) = \\ &= x - i\alpha^2 + \frac{1}{2}\frac{y^2+z^2}{x-i\alpha^2} + \cdots\ . \end{aligned}$$

If in the formula for G we substitute R by the latter expansion, preserving the two terms under the exponent (because of the presence of ω!) and only the first term in the amplitude, we shall obtain some approximation to that exact solution

$$G \simeq \frac{e^{\frac{\omega}{C}\alpha^2}}{4\pi} \cdot \frac{\exp\left\{i\frac{\omega}{C}\left[x + \frac{1}{2}\frac{y^2+z^2}{x-i\alpha^2}\right]\right\}}{x - i\alpha^2}$$

which is exactly a Gaussian beam propagating along the x-axis. Indeed, this expression coincides with the formula for a Gaussian beam if we take into consideration that in this case

$$\Gamma_{11} = \Gamma_{22} = \frac{1}{x - i\alpha^2} = \frac{x}{x^2+\alpha^4} + i\frac{\alpha^2}{x^2+\alpha^4}$$

and $\sqrt{\det Q} = x - i\alpha^2$.

This example shows that Gaussian beams can sometimes be deduced from some exact solutions under special conditions.

8.6 Gaussian beams in a medium with smooth interfaces

The problem of reflection and transmission of a Gaussian beam on a smooth interface has a lot in common with the corresponding problem of the ray method. In fact, even analytical formulas for Gaussian beams are very similar to those known in the paraxial ray theory. This enables us to use both our previous technique and some results for solving the reflection/transmission problem for Gaussian beams.

Our main goal now is to describe a procedure for extending the two complex valued solutions of the equations in variations along the reflected and transmitted central rays and to prove that the extended solutions preserve fundamental properties of the Gaussian beams.

The corresponding procedure is the following.

1) At a point of incidence of the incident Gaussian beam central ray we must construct the reflected and transmitted central rays for the corresponding reflected and transmitted Gaussian beams. Normally all types of rays should be taken into account, i.e. P and S rays.

2) Let us denote by T the 'eikonal' of a Gaussian beam, i.e.

$$T = \tau_o(s) + \frac{1}{2}\sum_{j,k=1}^{2}\Gamma_{jk}q_j q_k \, .$$

On the interface S we impose the following conditions on the eikonals

$$T_{in}\big|_S = T_r^{(a)}\big|_S = T_r^{(b)}\big|_S = T_{tr}^{(a)}\big|_S = T_{tr}^{(b)}\big|_S \qquad (8.23)$$

which should be accurately satisfied up to the second order terms with respect to q_1 and q_2. Evidently, equations (8.23) give rise to the linear relationship between matrices $\mathbf{\Gamma}$ for the incident and the reflected (transmitted) Gaussian beams (see Section 6.3 in Chapter 6). Then we add to equations (8.23) a requirement for q-components of the complex valued solutions $X^{(j)}$ to be continuous at the incident point. Both the latter requirement and equations (8.23) give rise to the following results at the incident point $s = s_*$

$$\begin{matrix} X_r^{(j)}(s_*) = \mathbf{M}_r X_{in}^{(j)}(s_*) \\ X_{tr}^{(j)}(s_*) = \mathbf{M}_{tr} X_{in}^{(j)}(s_*) \end{matrix} \quad j = 1,2 \qquad (8.24)$$

where $\mathbf{M}_r$ and $\mathbf{M}_{tr}$ are 4 x 4 matrices which describe (in linear approximation!) reflection and transmission of the rays from the ray tubes around the central rays.

It can be verified that $\mathbf{M}_r$ and $\mathbf{M}_{tr}$ both for P and S rays are sympletic matrices and, therefore the $\mathbf{J}$-scalar product of these complex solutions $X^{(j)}$ will be preserved at the incident point. This yields that the fundamental properties of Gaussian beams are preserved after reflection and refraction.

3) If ϕ_{in} means the initial "amplitude" of an incident Gaussian beam then for the initial "amplitudes" of the reflected and refracted Gaussian beams we obtain the following linear relationship

$$\phi_r = R_r\phi_{in} \, , \quad \phi_{tr} = R_{tr}\phi_{in} \, , \qquad (8.25)$$

where reflection R_r and transmission R_{tr} coefficients coincide, essentially, with the corresponding coefficients in the ray theory.

Conclusion. If the incident angle of the central ray of the incident Gaussian beam is such that no critical angles appear (both for P and S rays!) then, by using the algorithm described above, we satisfy approximately the boundary conditions on the interface in a vicinity of the incident point by constructing the reflected and refracted Gaussian beams. Note that the latter ones are uniquely constructed.

8.7 Gaussian beam integral

Point source problem.

In this case we have initially the central ray field, i.e. a family of rays emanated from the source in all directions, and the spherical angles θ, φ can be taken as the ray parameters. We consider each ray of the central ray field as the central ray $\vec{r} = \vec{r}_o(s, \theta, \varphi)$ and construct a Gaussian beam propagating along this ray. Let us denote it by $\vec{U}(s, q_1, q_2; \theta, \varphi)$. We then, obviously, get

$$\vec{U}(s, q_1, q_2; \theta, \varphi) = \frac{1}{\sqrt{\frac{1}{a_o}\det \mathbf{Q}}} \frac{\vec{t}}{\sqrt{a_o^2 \rho_o}} \exp\left[i\omega \big[\tau_o(s) + \frac{1}{2} \sum_{j,k=1}^{2} \Gamma_{jk} q_j q_k \big] \right] \quad (8.26)$$

for the case of a P-wave. Note that we consider here const=1.

Let us denote by $\phi_o(\theta, \varphi)$ some initial amplitude of the Gaussian beam (8.26) and construct the following integral over all rays from the central ray field

$$\vec{U} = \int_o^{\pi} d\theta \int_o^{2\pi} d\varphi \, \phi_o(\theta, \varphi) \vec{U}(s, q_1, q_2; \theta, \varphi) \, . \quad (8.27)$$

It is clear, that $\vec{U}$ given by equation (8.27) satisfies approximately the elastodynamic equations. But in order to describe precisely the wave field irradiated by the point source we have to find a suitable value for the initial amplitudes $\phi_o(\theta, \varphi)$. We shall discuss this problem in the next section.

Arbitrary initial data.

Certainly, we can use the Gaussian beam method in more general cases and not only for point sources.

However, the initial data cannot be arbitrary. They have to preserve peculiarities of the high frequency wave fields, that is, the wave front and distribution of the amplitude along the wave front have to be given separately. In other words, we may say that the initial data must have the form of a ray series, at least of the leading term of a ray series, in order to propagate them by means of the Gaussian beam method.

Suppose we know an initial wave front $\tau = t(= \text{const})$. Then we can construct a family of rays orthogonal to the wave front and some coordinates α, β on this wave front can be taken as the ray parameters. Next, we construct a Gaussian beam for each ray from this family of rays. Let us denote it by $\vec{U}(s, q_1, q_2; \alpha, \beta)$ where α, β are the ray parameters. Further, we introduce the so far unknown initial amplitudes $\phi_o(\alpha, \beta)$ of the Gaussian beams and present the wave field as an integral

$$\vec{U} = \iint\limits_{(\tau=t)} d\alpha d\beta \phi_o(\alpha, \beta) \vec{U}(s, q_1, q_2; \alpha, \beta) \quad (8.28)$$

over the whole surface of the wave front $\tau = t$. This integral, clearly, satisfies approximately the elastodynamic equations, but in order to describe exactly the wave field caused by given initial data, the initial amplitudes $\phi_o(\alpha, \beta)$ should be specifically chosen. Note that so far we have not referred to distribution of the amplitude on the wave front.

Along with integral (8.28) we can extend initial data by means of the ray method. Indeed, in some vicinity of $\tau = t$ the family of rays will be regular and this fact enables us to get the ray method formulas for the wave field. This implies that we can obtain the amplitude and the eikonal in some vicinity of the initial wave front. And it is enough to find ϕ_o.

8.8 Initial amplitudes $\phi_o(\alpha, \beta)$ for the Gaussian beam integral

Let us insert expression (8.26) for a Gaussian beam in integral (8.27) or (8.28) in order to illustrate the analytical nature of the corresponding double integral in 3D. Evidently, in 2D we obtain a single integral because in this case the family of rays depends only upon one ray parameter. Thus, in the case of P-waves we get the following formula

$$\vec{U} = \int_o^{2\pi} d\varphi \int_o^{\pi} d\theta A \exp(i\omega f)\vec{t}\,, \tag{8.29}$$

where

$$f = \tau_o(s) + \frac{1}{2}\sum_{j,k=1}^{2} \mathbf{\Gamma}_{jk} q_j q_k \,;\; A = \frac{\phi_o(\theta, \varphi)}{\sqrt{a_o(s)\rho_o(s)\det \mathbf{Q}}} \tag{8.30}$$

and frequency ω is the large parameter. It follows from equations (8.29) and (8.30) that we get an integral with quick oscillating integrand and therefore asymptotics of the integral as $\omega \to \infty$ can be derived by means of the stationary phase method, or being more precise, by means of some extension of this method because $\mathbf{\Gamma}$ is a complex-valued matrix.

It turns out that in a domain where the family of rays involved in the integral is regular, the high frequency asymptotics of the integral has precisely the form of a ray series. It holds true both for the point source problem and for the arbitrary case as well.

On the other hand, in the domain where the family of rays is regular we can obtain the ray method formula for the wave field and this formula is unique and well-defined.

Then, by comparing both these expressions for the wave field we obtain the final formula for initial amplitudes ϕ_o.

Unfortunately, all necessary calculations are not simple due to the fact that all functions of the ray parameters in integral (8.29) are given in an implicit form. In some particular cases they can be simplified.

Only the most important steps on that way are described. For more details, see Popov (1981,1982).

a) Consider the point of observation M located in a domain where the family of rays is regular. Hence, there is only one ray passing through it. Let it be the ray given by equations $\theta = \theta_o, \varphi = \varphi_o$ and we denote it by $\vec{r}_o(s)$.

   It is clear that the main contribution at M is given by the Gaussian beam having this ray $\vec{r}_o(s)$ as the central one. Contributions from others Gaussian beams to the point are smaller due to an exponential decrease of all of them with an increase in distance to M.

   This leads to the fact that the stationary point is $\theta = \theta_o$, $\varphi = \varphi_o$ and that it is a simple (non-degenerate) stationary point.

b) On the next step we have to expand the function f – see equation (8.30) – in power series on $\theta - \theta_o$ and $\varphi - \varphi_o$. This means that now we have to take into account contributions from all other Gaussian beams from the vicinity of the ray $\vec{r}_o(s)$ to the integral.

   Let us denote by s', q_1', q_2' the ray centered coordinates referred to the ray $\vec{r}_o$ (so now we write $\vec{r}_o(s')$!).

   It is possible to deduce by studying the connecting formulas between s, q_1, q_2 and s', q_1', q_2' that

$$\begin{aligned} \left.\frac{\partial q_j}{\partial \theta}\right|_{\theta=\theta_o,\varphi=\varphi_o} &= -\left.\frac{\partial q_j'}{\partial \theta}\right|_{\theta=\theta_o,\varphi=\varphi_o} , \\ \left.\frac{\partial q_j}{\partial \varphi}\right|_{\theta=\theta_o,\varphi=\varphi_o} &= -\left.\frac{\partial q_j'}{\partial \varphi}\right|_{\theta=\theta_o,\varphi=\varphi_o} . \end{aligned} \tag{8.31}$$

   Further, eikonal $\tau_o = \int (1/C_o(s))ds$ along the central ray of the Gaussian beam being calculated for all rays of the family of rays under consideration defines a global function in the domain where the family of rays is regular. And this function can be investigated by means of the paraxial ray theory. Hence, in a vicinity of the ray $\vec{r}_o(s')$ we get

$$\tau = \tau_o(s') + \frac{1}{2}\sum_{j,k}^{2} G_{jk} q_j' q_k' + \cdots . \tag{8.32}$$

   Note that in order to avoid misunderstandings we denote by $\mathbf{G}$ the matrix in power series (8.32) for the eikonal τ and preserve $\mathbf{\Gamma}$ for the Gaussian beams.

   If now $\tau = t_M$ is precisely the wave front on which an observation point M is located, then the equation of this wave front with accuracy up to the terms of second order reads

$$\tau_o = t_M - \frac{1}{2}\sum_{j,k=1}^{2} G_{jk} q_j' q_k' + \cdots . \tag{8.33}$$

Then, following the paraxial ray theory we can present it in the form

$$\mathbf{G} = \mathbf{G}_p \mathbf{G}_q^{-1} \tag{8.34}$$

instead of $\mathbf{PQ}^{-1}$ but with the same sense as $\mathbf{P}$ and $\mathbf{Q}$ (see Chapter 5). Thus, we obtain the following expression for the function f in equation (8.29)

$$f = t_M + \frac{1}{2} \sum_{j,k=1}^{2} (\mathbf{\Gamma} - \mathbf{G})_{jk} q'_j q'_k + \cdots \, . \tag{8.35}$$

The last remark to this point should be that only coordinates q'_j, $j = 1, 2$ have to be differentiated with respect to the ray parameters, though $\mathbf{\Gamma}$ depends on the ray parameters too. It follows from this fact that for $\theta = \theta_o$, $\varphi = \varphi_o$ we have $q'_1 = q'_2 = 0$ and therefore the derivatives of $\mathbf{\Gamma}$ vanish! And the following derivatives

$$\left.\frac{\partial q'_j}{\partial \theta}\right|_{\theta_o \varphi_o} , \quad \left.\frac{\partial q'_j}{\partial \varphi}\right|_{\theta_o \varphi_o} , \; j = 1, 2 \, ,$$

calculated on the central ray $\vec{r}_o(s')$ form matrix $\mathbf{G}_q$ which is responsible for the geometrical spreading. In fact, $J = |\det \mathbf{G}_q|$ in our case!

Finally we get the following result

$$f = t_M + \frac{1}{2} \sum_{j,k=1}^{2} \left(\mathbf{G}_q^T (\mathbf{\Gamma} - \mathbf{G}) \mathbf{G}_q \right)_{jk} \alpha_j \alpha_k + \cdots \, , \tag{8.36}$$

where $\alpha_1 = \theta - \theta_o$, $\alpha_2 = \varphi - \varphi_o$.

c) According to the stationary phase method for double integrals we get the following expression for the main term of the asymptotics (after introducing new variables $\beta_1 = \sqrt{\omega/2}(\theta - \theta_o)$ and $\beta_2 = \sqrt{\omega/2}(\varphi - \varphi_o)$)

$$\vec{U}_{(M)} \simeq \frac{2}{\omega} A(s_M) e^{i\omega t_M} \, \vec{t}(s_M) \cdot I \, , \tag{8.37}$$

where $A(s_M)$ means the amplitude A – see (8.30) – calculated on the central ray $\theta = \theta_o$, $\varphi = \varphi_o$ at the point of observation $s = s_M$. Integral I in equation (8.37) takes the form

$$I = \iint\limits_{-\infty}^{+\infty} \exp \left\{ i \sum_{j,k=1}^{2} (\mathbf{G}_q^T (\mathbf{\Gamma} - \mathbf{G}) \mathbf{G}_q)_{jk} \beta_j \beta_k \right\} d_{\beta_1} d_{\beta_2}$$

and can be calculated exactly by developing the quadratic form to the sum of squares. Note that it converges because of the presence of $\operatorname{Im} \mathbf{\Gamma}$.

We eventually obtain on the right-hand side of equation (8.37) the following expression

$$\vec{U}_{(M)} \simeq \frac{2\pi i}{\omega} \frac{\phi_o(\theta_o, \varphi_o)}{\sqrt{\det \mathbf{L}(0)}} \frac{\exp(i\omega t_M)}{\sqrt{a_o(s_M)\rho_o(s_M) \det \mathbf{G}_q(s_M)}} \vec{t}(s_M) \tag{8.38}$$

where $\mathbf{L}(0)$ is some matrix, which does not depend upon s.

Apparently, formula (8.38) has the form of the zero-order term of a ray series with the eikonal $\tau = t_M$ and the geometrical spreading $J = |\det \mathbf{G}_q(s_M)|$. This enables us to find $\phi_o(\theta_o, \varphi_o)$ by comparing it with the ray method asymptotics of the wave field under consideration.

8.9 Numerical algorithm of the Gaussian Beam method

An algorithm for computation of the wave field by means of the Gaussian beam method can be divided into the following three steps.

i) In a vicinity of an observation point M we have to construct a fan of rays (or a ray diagram) which covers this vicinity more or less uniformly.

ii) We have to construct a Gaussian beam propagating along the central ray for each ray of the fan.

iii) The contribution of all Gaussian beams to the wave field has to be summarized at the observation point M.

Consider each step separately.

i) In order to construct the ray diagram in a vicinity of M we can use well-known algorithms in the ray theory. The new moment consists in constructing the ray centered coordinates for each ray from the fan. To this end we have to solve the differential equations for one of the vectors $\vec{e}_j$ along the ray in case of a 3D medium (the second one can be found by means of algebra). Note that in 2D we are not obliged to solve additional equations for $\vec{e}$. Then we have to find ray centered coordinates of an observation point M for each ray.

ii) To construct a Gaussian beam we have to know two complex valued solutions of the equations in variations (in 3D). Actually, it implies that a full system of linearly independent solutions of the equations should be known, or in other words, the fundamental matrix of the equations in variations. And it is more convenient to work with this matrix, say $\mathbf{W}(s)$. If we know it, we can retrieve the complex valued solutions $X^{(j)}(s)$ by the formula $X^{(j)}(s) = \mathbf{W}(s)Z^{(j)}$, where $Z^{(j)}$ are appropriate complex initial values for $X^{(j)}$ (for example, at the point source). Note that in a 2D case the fundamental matrix $\mathbf{W}$ is formed by two linearly independent solutions.

On an interface, an incident Gaussian beam gives rise to the reflected and the refracted Gaussian beams. In order to construct them, the reflection (refraction) matrices for the rays close to the central ones should be involved along with the reflection (refraction) coefficients for the amplitudes of the Gaussian beams.

iii) To summarize the contributions of all Gaussian beams to the wave field at the observation point, we can use any approximate formulas for numerical integration. The final result will depend upon the density of rays in the fan. But it seems relevant to mention here that (a) we do not need to construct the ray passing exactly through the observation point and (b) by using one fan of rays we can compute the wave field at many observation points.

In order to minimize the contribution of Gaussian beams distant from M it seems natural to use such values of parameters γ_1, γ_2 that provide a narrow width of Gaussian beams in the vicinity of M.

8.10 On the role of the Gaussian beam free parameters

In order to illustrate the role of the free Gaussian beam parameters, consider the point source problem in a 2D homogeneous medium

$$(\Delta + k^2)U = \delta(x)\delta(y) .$$

The exact solution of the problem reads $U = -(i/4)H_o^{(1)}(kr)$. We have the following asymptotics for U as $kr \to \infty$

$$U = -\frac{i}{4}\sqrt{\frac{2}{\pi kr}}e^{i(kr-\frac{\pi}{4})}\left[1 - \frac{i}{8kr} + O\left(\frac{1}{(kr)^2}\right)\right] . \tag{8.39}$$

On the other hand we can apply the Gaussian Beam method to the problem, if k is assumed to be large. In this case we have the central ray field with its center located at the origin $x = y = 0$, and the polar angle φ is the ray parameter for the family of rays. Denote by $U_\varphi(s,n)$ a Gaussian beam propagating along the ray fixed by the angle φ, then

$$U_\varphi(s,n) = \frac{1}{\sqrt{a_1 + ia_2 s}} \exp\left[ik\left(s + \frac{1}{2}\frac{ia_2}{a_1 + ia_2 s}n^2\right)\right] \tag{8.40}$$

where a_1 and a_2 are arbitrary parameters of the Gaussian beam (they both must be positive to provide Im $(P/Q) > 0$).

The integral $\tilde{U}$ over the Gaussian beams has the form

$$\tilde{U} = \int_o^{2\pi} \phi_o(\varphi)U_\varphi(s,n)d\varphi, \quad \phi_o(\varphi) = -\frac{i}{4\pi}\sqrt{a_1} . \tag{8.41}$$

In order to estimate precision of the approximate solution $\tilde{U}$, we have to investigate the absolute value $|U - \tilde{U}|$ as a function of a_1, a_2. It is not so easy to analytically estimate the difference $|U - \tilde{U}|$, so we will proceed as follows. Assume, that a_1 and a_2 are the same for all the rays, i.e. they are independent of φ, and let us calculate the asymptotics of the integral $\tilde{U}$ as $k \to \infty$ within the accuracy up to the second term of the stationary phase method. By denoting r_o the distance to an observation point, we get after some calculations

$$\tilde{U} = -\frac{i}{4}\sqrt{\frac{2}{\pi k r_o}} e^{i\left(k r_o - \frac{\pi}{4}\right)} \left[1 - \frac{i}{8kr_o}\left(1 - 3r_o^2 \frac{a_2^2}{a_1^2}\right) + O(k^{-2})\right] . \tag{8.42}$$

By comparing the second item in equation (8.39) to (8.42) we come to the following conclusions:

i) If $r_o, a_2/a_1$ are fixed and $k \to \infty$ the second item will be as small as we wish and the influence of the multiplier $\left(1 - 3r_o^2(a_2^2/a_1^2)\right)$ will be small. As this turns out, the Gaussian beam integral $\tilde{U}$ provides the asymptotics of the exact solution U with respect to the wave number k.

ii) On the contrary, if k is fixed and we increase the distance r_o between the source and the receiver the second term in equation (8.42) will increase due to the multiplier $\left(1 - 3r_o^2(a_2/a_1)^2\right)$. Hence, precision of asymptotic solution $\tilde{U}$ decreases. This means that, in general, the Gaussian beam asymptotics is not uniform with respect to the distance!

Now let us choose the arbitrary parameters a_1, a_2 in such a way that the width of the Gaussian beam in a vicinity of the observation point is as small as possible. To this end consider $Im\mathbf{\Gamma}$ in equation (8.40)

$$\begin{aligned} \mathbf{\Gamma} &= \frac{ia_2(a_1 - ia_2 s)}{(a_1 + ia_2 s)(a_1 - ia_2 s)} = \frac{ia_1 a_2}{a_1^2 + (a_2 s)^2} + \frac{a_2^2 s}{a_1^2 + (a_2 s)^2} . \\ \text{Im}\, \mathbf{\Gamma} &= \frac{a_1 a_2}{a_1^2 + (a_2 s)^2} = \frac{\varepsilon}{1 + \varepsilon^2 s^2} ; \quad \varepsilon = \frac{a_2}{a_1} . \end{aligned}$$

By differentiating $\text{Im}\,\Gamma$ with respect to ε and putting $(d/d\varepsilon)\text{Im}\,\Gamma = 0$, we obtain consistently

$$\frac{d}{d\varepsilon}\text{Im}\,\Gamma = \frac{1 - \varepsilon^2 s^2}{(1 + \varepsilon^2 s^2)^2} = 0 \Rightarrow \varepsilon^2 = \frac{1}{s^2} .$$

This means that the minimum of the width of the Gaussian beam takes place if $\varepsilon = 1/r_o$, where r_o is the distance between the source and the observation point. If we assume further that $a_2/a_1 = 1/r_o$ the second term in equation (8.42) takes the form

$$\frac{i}{8kr_o}\left(1 - 3r_o^2 \frac{a_2^2}{a_1^2}\right) = -2\frac{i}{8kr_o}$$

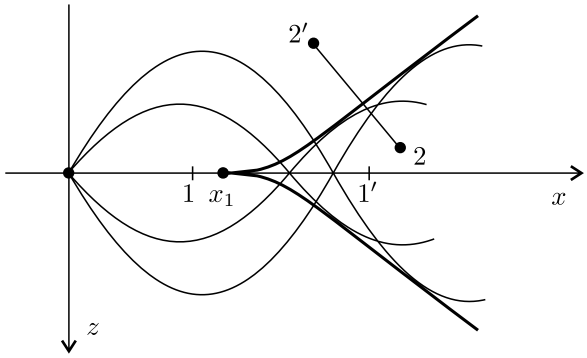

Figure 8.1: Structure of rays and caustics in the wave guide. A point source is located at the origin.

and, hence, the error provided by the integral $\tilde{U}$, (8.41), and caused by the second term in equation (8.42), does not increase when the distance to the source increases.

Thus, this simple example shows that with a fixed wave number, the region of validity of the Gaussian beam method can be extended by means of a special selection of the arbitrary parameters. However, so far the question of an optimum selection of such parameters is not entirely clear.

8.11 Numerical examples

We consider a wave guide formed by the velocity model $a = a_1 + \alpha z^2$. The axis of the wave guide coincides with the horizontal x-axis. Behavior of rays is depicted in Fig. 8.1. The caustic is formed by two symmetrical branches of a curve (bold lines on Fig. 8.1) and has a cusp located at the point $x = x_1$.

Example 1

On Fig. 8.2, the modulus of P-wave $|\vec{U}|$ is calculated on interval $1 - 1'$, which includes the cusp of the caustic, by means of the Gaussian beam method (solid line) and the ray method (dashed line). The ray method fails in a vicinity of the cusp $x = x_1$ but both methods provide a good coincidence apart from the interval $|x - x_1| \leq 1,5\lambda_a$, where λ_a is the wave length of P-wave.

Example 2

On Fig. 8.3 the wave field is computed on the interval $2 - 2'$ which intersects the upper branch of the caustic under the angle $\frac{\pi}{2}$ (see Fig. 8.1). Note that only the rays reaching the caustic from below are taken into account in the calculations. We can observe a typical transition of the wave field from the light where the wave field is oscillating to the caustic shadow where it decays exponentially. Note that the focusing phenomenon on the caustic becomes stronger with the increase in the frequency of the wave field.

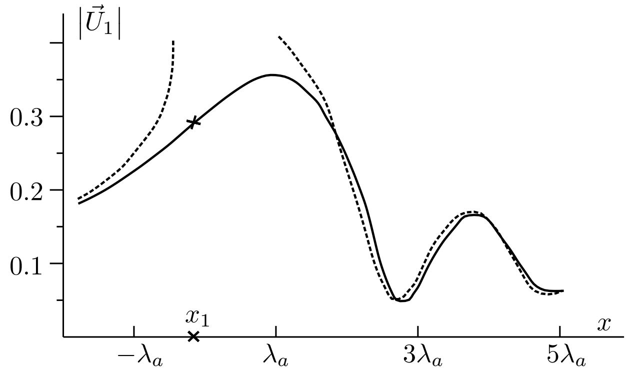

Figure 8.2: Modulus of the P-wave in a vicinity of the cusp of the caustic computed by the ray method (dashed line) and by the Gaussian Beam method (solid line).

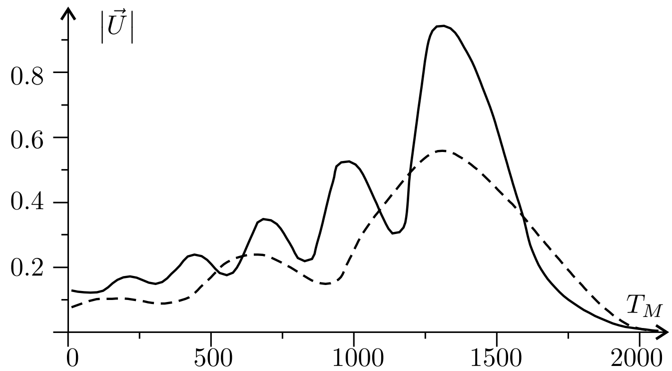

Figure 8.3: Modulus of the P-wave computed in a vicinity of a simple branch of the caustic for $10Hr$ (solid line) and $5Hr$ (dashed line).

Example 3 Reflection from a concave traction free boundary.
Equation of the boundary is taken in the following form

$$Z = R\left[\exp\left(-\frac{1}{2}\frac{X^2}{R^2}\right) - 0,1\right], \tag{8.43}$$

$$R = 20 \text{ km}. \tag{8.44}$$

A point source – center of dilatation – is placed at the origin. We consider a homogeneous medium, $a = 4000$ m/sec, $b = 2500$ m/sec. The incident P-wave $\vec{U}_{in}^{(p)}$ with frequency $\omega = 20Hr$ excites on the boundary reflected S-wave $\vec{U}_r^{(s)}$ and

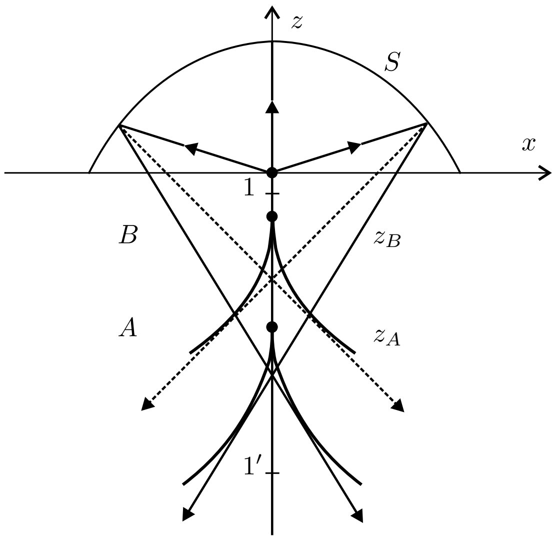

Figure 8.4: Structure of rays and caustics after reflection from a concave boundary. P-rays are depicted by solid lines and S-rays by dashed lines.

reflected P-wave $\vec{U}_r^{(p)}$ which have caustics B and A, respectively, (bold lines on Fig. 8.4). The total reflected wave field $|\vec{U}| = |\vec{U}_r^{(p)} + \vec{U}_r^{(s)}|$ is depicted in Fig. 8.5 (line 1) and $|\vec{U}_r^{(s)}|$ is presented by line 2. The wave field is calculated by means of the Gaussian beam method on an interval which contains both cusp points Z_B for S rays and Z_A for P rays.

In general, the behavior of the total wave field modulus is similar to the one depicted in Fig. 8.2 and is typical for the vicinity of a cusp point.

The behavior of $|\vec{U}_r^{(s)}|$ is not typical and line 2 on Fig. 8.5 follows the behavior of the reflection coefficient of the S wave. On the axis $x = 0$, it is equal to zero and grows monotonously with an increasing modulus of the incident angle.

It follows from these numerical experiments that the Gaussian beam method does not face problems on caustics with a different geometrical structure.

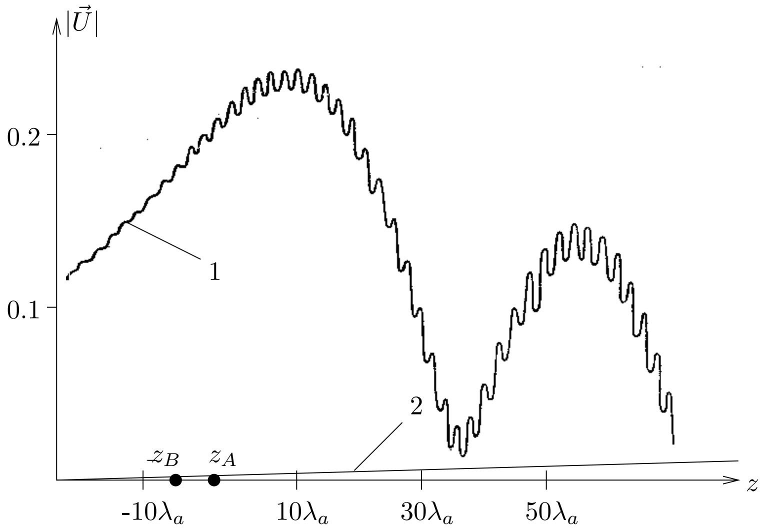

Figure 8.5: The wave field reflected by a concave traction free boundary. The incident wave field is generated by the center of dilatation.

9

The Gaussian Beam and the ray methods in the time-domain

9.1 Correlation between non-stationary and stationary wave fields

Wave propagation problems in elastodynamics in the time-domain can be formulated as follows. The displacement vector $\vec{U}$ satisfies a non-homogeneous system of elastodynamic equations

$$\hat{L}\vec{U} - \rho\frac{\partial^2\vec{U}}{\partial t^2} = \vec{F}(t, x, y, z) \tag{9.1}$$

where $\hat{L}$ means an operator containing derivatives with respect to the spatial variables (its explicit form is given in Chapter 7). Vector $\vec{F}$ has a sense of external forces, or a source of the wave field. We have to impose also an initial conditions for $\vec{U}$ at the initial moment, say, $t = 0$, the so-called Cauchy data $\vec{U}\big|_{t=0} = \vec{U}_o$ and $\partial\vec{U}/\partial t\big|_{t=0} = \vec{U}_{to}$. If we consider the wave field generated only by the source $\vec{F}$ we have to assume that $\vec{U}_o = 0$ and $\vec{U}_{to} = 0$, or in other words, impose that $\vec{U} = 0$ for $t < 0$.

In geophysics the following point sources are used rather often. The center of dilation

$$\vec{F} = \vec{F}_1 \equiv f(t)\text{grad}\,(\delta(M - M_o))\,, \tag{9.2}$$

and the center of rotation

$$\vec{F} = \vec{F}_2 \equiv f(t)\text{rot}\,(\vec{l}\,\delta(M - M_o))\,, \tag{9.3}$$

where $\vec{l}$ is a constant unit vector and $f(t)$ is the temporal action or the wavelet. In geophysical applications $f(t)$ can be described as a pulse modulated both in amplitude and frequency

$$f(t) = Re\{A(t)e^{ip\theta(t)}\}\ , \tag{9.4}$$

where $A(t)$ and $\theta(t)$ are sufficiently smooth functions and $A(t)$ is substantially different from zero on a finite interval of time. The derivative of $\theta(t)$ does not vanish on this interval and describes an instantaneous frequency $\omega(t) = -p\theta'(t)$ (we assume that $\theta'(t) < 0$). Now, if p is supposed to be large, we get quick oscillations because of $\exp(ip\theta)$ and a smooth envelope given by $A(t)$.

Let us denote by $f_F(\omega)$ the Fourier transform of $f(t)$

$$f_F(\omega) = \int_{-\infty}^{+\infty} e^{i\omega t} f(t)dt\ . \tag{9.5}$$

Suppose further that we know the solution of the stationary problem

$$\hat{L}\vec{G} + \rho\omega^2\vec{G} = \left\{ \begin{array}{l} \text{grad } (\delta(M - M_o)) \\ \text{rot } (\vec{l}\,\delta(M - M_o)) \end{array} \right.\ , \tag{9.6}$$

then the solution of the non-stationary problem (9.1) can be presented in the form

$$\vec{U} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega t} f_F(\omega)\vec{G}(M, M_o; \omega)d\omega\ . \tag{9.7}$$

Hence, if we are able to construct the solution of stationary problem (9.6) in the frequency domain, we shall solve non-stationary problem (9.1) by means of formula (9.7).

Let us prove that the integral in equation (9.7) can be reduced to the integral over $0 \le \omega < +\infty$, i.e.

$$\vec{U} = \frac{1}{\pi}\ \text{Re}\ \int_{o}^{\infty} e^{-i\omega t} f_F(\omega)\vec{G}(M, M_o, \omega)d\omega \tag{9.8}$$

for the real solution of the initial problem (9.1).

Proof. For the real solution we have

$$\vec{U} = \overline{\vec{U}} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{i\omega t}\overline{f_F}(\omega)\overline{\vec{G}}(M, M_o; \omega)d\omega\ .$$

By introducing a new variable of integration $\omega' = -\omega$ we obtain

$$\vec{U} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega' t}\overline{f_F}(-\omega')\overline{\vec{G}}(M, M_o; -\omega')d\omega'\ . \tag{9.9}$$

By comparing the integrals in equations (9.7) and (9.9) we get

$$\overline{f_F}(-\omega')\overline{\vec{G}}(M, M_o; -\omega') = f_F(\omega')\vec{G}(M, M_o; \omega')\ . \tag{9.10}$$

Consider now the integral over $(-\infty, 0)$ in formula (9.7). It can be developed as follows

$$\int_{-\infty}^{0} e^{-i\omega t} f_F(\omega)\vec{G}(M, M_o; \omega)d\omega = \int_{o}^{\infty} e^{i\omega' t} f_F(-\omega')\vec{G}(M, M_o; -\omega')d\omega'$$

$$= \int_{o}^{\infty} e^{i\omega' t}\overline{f_F}(\omega')\overline{\vec{G}}(M, M_o; \omega')d\omega' = \overline{\int_{o}^{\infty} e^{-i\omega' t} f_F(\omega')\vec{G}(M, M_o; \omega')d\omega'}$$

due to equation (9.10) holding true (note that we actually used an equation which is a complex conjugate with respect to (9.10)).

Taking into account the latter result we obtain

$$\begin{aligned}\vec{U} &= \frac{1}{2\pi}\int_{o}^{\infty} e^{-i\omega' t} f_F(\omega')\vec{G}(M, M_o; \omega')d\omega' + \\ &+ \overline{\frac{1}{2\pi}\int_{o}^{\infty} e^{-i\omega' t} f_F(\omega')\vec{G}(M, M_o; \omega')d\omega'} = \\ &= \frac{1}{\pi} Re \int_{o}^{\infty} e^{-i\omega' t} f_F(\omega')\vec{G}(M, M_o; \omega')d\omega'\end{aligned}$$

which is exactly formula (9.8).

Thus, in order to construct the real solution for a non-stationary problem (9.1) we have to know the solution for the corresponding stationary problem (9.6) only for positive frequency ω.

Let us dwell on some geometrical peculiarities of wave propagation caused by the point source problem (9.1), (9.2) or (9.1), (9.3). We assume that $\vec{U} \equiv 0$ for $t < 0$ and therefore the wave field is generated only by the source. For the sake of simplicity, consider a 2D homogeneous medium.

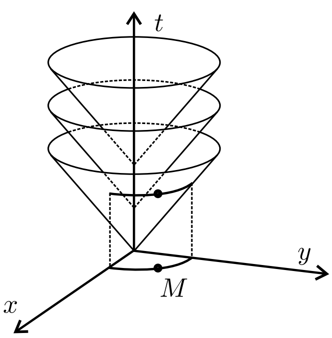

Figure 9.1: Scheme of wave propagation in the space-time domain caused by a point source located at the origin.

In order to present the wave field in the space-time domain x, y, t, we have to draw a set of cones emanated from the source at each moment $t_0 \in (0, T)$, where T means the duration of the temporal action $f(t)$. These cones are called characteristics and the wave field propagates along them. They themselves are formed by straight lines

$$t - t_o = \frac{s}{C}\,, \quad x = s\cos\varphi\,, \quad y = s\sin\varphi\,; \varphi \in (0, 2\pi),\; t_o \in (0, T)\,, \tag{9.11}$$

where C is the velocity of the wave and s is the arc length along the lines. Straight lines (9.11) in the space-time t, x, y are called the bicharacteristics and their projections onto x, y space are called the rays.

Consider a plane $t = t_1$ in space-time orthogonal to t-axis (obviously, it is parallel to the space x, y). This plane intersects each cone along a circle. The projection of the circle onto x, y space gives the position of the wavefront at the moment $t = t_1$. Thus, in the x, y-space we obtain a set of circular wavefronts emanated from the source at each moment $t \in (0, T)$.

In order to describe the wave field at an observation point M, we have to depict the straight line starting at M in an orthogonal direction to the x, y - plane. The wave field reaches M at the moment when this line intersects the first cone which was emanated from the source at the moment $t = 0$.

This picture becomes more and more complicated with the increase in time because each interface of the medium will give rise to the reflected and transmitted waves and because more extended domains will be involved into the process of propagation of waves.

9.2 The Fourier transform: high frequency and smoothness

Consider some features of the Fourier transformation:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega t} f_F(\omega) d\omega\,. \tag{9.12}$$

The integral (9.12) will exist if

$$\int_{-\infty}^{+\infty} |f_F(\omega)|\, d\omega \le \text{const} < \infty\,.$$

i) Assume first, that $f_F(\omega) \ne 0$ if $\omega \in [a, b]$ and $f_F(\omega) \equiv 0$ for all other values of ω. In this case we have

$$f(t) = \frac{1}{2\pi}\int_a^b e^{-i\omega t} f_F(\omega) d\omega$$

and $f(t)$ is a smooth function of t.

Indeed, by differentiating n times the function $f(t)$ with respect to t we obtain

$$\frac{d^n}{dt^n} f(t) = \frac{1}{2\pi}\int_a^b e^{-i\omega t} (-i\omega)^n f_F(\omega) d\omega$$

and the integral on the right-hand side does exist because of finite limits of integration! This means that $f(t) \in C^{\infty}$, i.e. it can be differentiated an arbitrary number of times.

ii) Assume now that the asymptotics of $f_F(\omega)$ as $\omega \to \infty$ has the form

$$f_F(\omega) = \frac{\text{const}}{\omega^n}\ , \ n \text{ is integer}\ , \quad n > 1,$$

and let us observe the smoothness of the function $f(t)$ in this case. To this end we have to investigate only the integral over the interval (Ω, ∞) where Ω is supposed to be large enough so that $f(t)$ can be substituted in the integral by its asymptotics. By differentiating $(n-1)$ times function $f(t)$ we arrive at the following integral to be studied

$$\int_{\Omega}^{\infty} e^{-i\omega t}(-i\omega)^{n-1}\frac{\text{const}}{\omega^n}d\omega\ , \quad \Omega \gg 1\ .$$

One can verify just by integrating in parts that the latter integral converges for $t \neq 0$ and does not converge if $t = 0$. This means that the $(n-1)$ order derivative of function $f(t)$ is singular at $t = 0$ while the derivatives of a lesser order are smooth.

Hence, in this case we obtain the following result: the asymptotic behavior of $f_F(\omega)$ for large ω is responsible for the smoothness of the temporal action $f(t)$ and singularities may appear when $t = 0$.

9.3 The ray method in the time domain

A ray method series in the ω-domain has the form

$$\vec{U} = e^{i\omega\tau}\sum_{k=0}^{\infty}\frac{\vec{U}_k}{(-i\omega)^k}\ . \tag{9.13}$$

One of the ways to develop the ray method in the time domain is simply to replace $\vec{G}(M, M_o; \omega)$ in equation (9.7) by its ray asymptotics (9.13) which is valid for large ω, and carry out the integration over ω just formally (in fact, this integration is being done in terms of distributions).

Consider in more detail the main term (or zero-order term) of the series (9.13). For the main term $\vec{U}_o(M, M_o; t)$ we get:

$$\vec{U}_o(M, M_o; t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-i\omega t + i\omega\tau} f_F(\omega)\vec{U}_o d\omega\ . \tag{9.14}$$

If $\vec{U}_o$ does not depend upon ω we obtain immediately

$$\vec{U}_o(M, M_o; t) = \vec{U}_o f(t - \tau)\ , \tag{9.15}$$

where f is the temporal action at the source.

A corresponding procedure with the second term leads to the following expression for $\vec{U}_1(M, M_o; t)$

$$\vec{U}_1(M, M_o; t) = \vec{U}_1 g(t - \tau)$$

where $g'(y) \equiv dg(y)/dy = f(y)$ because of the presence of $-i\omega$ in the denominator!

Thus, we obtain the following ray method series in the time domain

$$\vec{U}(M, M_o; t) = \sum_{n=0}^{\infty} \vec{U}_n(M, M_o) F^{(n)}(t - \tau) \,, \tag{9.16}$$

where the set of functions $F^{(n)}(y)$ satisfies the following conditions

$$\frac{d}{dy} F^{(n)}(y) = F^{(n-1)}(y) \,,\ n = 1, 2, \cdots .$$

Now we may forget the ray series (9.13) in the ω-domain and work with the expression (9.16) just by inserting it directly into the elastodynamic equations. Expression (9.16) is called the ray method expansion in the time domain.

But now it is clear that since equation (9.13) holds true only for the large frequency, formula (9.15) and respectively series (9.16) contain a systematic error due to asymptotics (9.13) having been extended somehow on a small ω during the process of integration. Hence, along with an approximate expression (9.16) for the wave field, we should have some background. Evidently, such background is a smooth function of t (see previous section), and it does not influence jumps of the wave field precisely on the wavefront $t = \tau$.

Taking into account all these considerations we may say that the ray method in the time domain is the asymptotics of the wave field with respect to smoothness.

9.4 The Gaussian Beam method in the time domain

A standard way of employing the Gaussian beam method for non-stationary problems is to replace the function $\vec{G}(M, M_o; \omega)$ in equation (9.7) by the integral over Gaussian beams. In this case it is necessary to compute the Fourier transform for every Gaussian beam that contributes to the wave field at an observation point M. This is precisely what a number of papers in geophysics has used- see the review paper by Červený (1985) and Babich and Popov (1989).

There is a different approach based on the space-time Gaussian beams - see Popov (1987). A brief description of its main ideas is as follows.

We have constructed some asymptotic solutions of non-stationary elastodynamic equations which preserve the fundamental properties of Gaussian beams in the frequency domain: each solution is related to a bicharacteristic and has no singularities. Naturally, it depends upon time and space variables. These solutions are called the space-time Gaussian beams (or quasi-jets). We remind that in the case of a 2D homogeneous medium the bicharacteristics are straight lines

- see section 9.1 formula (9.11). Next we construct the asymptotics of the wave field in the time domain directly by integrating the space-time Gaussian beams over the ray parameters just as we do in the frequency domain. This method is adequate to situations when the wave field is formed by waves modulated both in the frequency and the amplitude and the wavelet is given by formula (9.4). For applications of the method see Kachalov and Popov (1988, 1990). Unfortunately, the method requires a rather detailed description and therefore we do not present it here.

APPENDIX A
Integrals of oscillating functions

Consider an integral

$$I = \int_a^b g(x) e^{i\omega f(x)} dx$$

where ω is supposed to be large, i.e. $\omega \to \infty$. Suppose, $g(x)$ and $f(x)$ are sufficiently smooth functions, i.e. they can be differentiated on the interval (a, b) as many times as we need.

Let us assume, that $f'(x) \neq 0$ for all x from the interval $[a, b]$. Then the asymptotics of I as $\omega \to \infty$ can be obtained by the integration in parts

$$I = \int_a^b e^{i\omega f(x)} \frac{g(x)}{f'(x)} df = \frac{e^{i\omega f(x)}}{i\omega} \left. \frac{g(x)}{f'(x)} \right|_{x=a}^{x=b} - \frac{1}{i\omega} \int_a^b e^{i\omega f(x)} \left(\frac{g(x)}{f'(x)} \right)' dx \, .$$

If we integrate the latter integral in parts again we shall conclude that it is of order $O\left(\omega^{-2}\right)$.

Thus, it follows from the above that the main term of the asymptotics of I for a large ω reads

$$I = \frac{1}{i\omega} \left[e^{i\omega f(b)} \frac{g(b)}{f'(b)} - e^{i\omega f(a)} \frac{g(a)}{f'(a)} \right] + 0 \left(\frac{1}{\omega^2} \right) \, .$$

A.1 Geometrical interpretation

By recollecting the geometrical sense of an integral as the area between the x-axis and the graph of the integrand, we can present the following interpretation of the result described above. Due to the fast oscillations of Re $(g(x)\exp(i\omega f(x))$ and Im $(g(x)\exp(i\omega f(x))$ as well, the areas with opposite signs compensate each other except for those two, which join the end points of integration. But with the

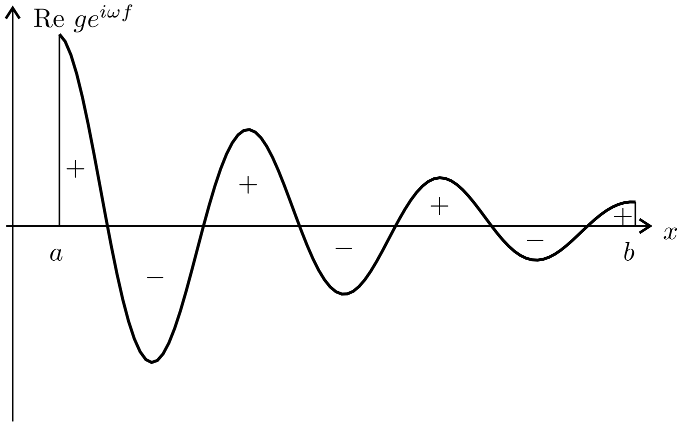

Figure A.1: Fast oscillations of the integrand cause compensation of areas with opposite signs.

increasing ω those areas tend to zero too. Thus, the whole integral is almost equal to zero for a large ω.

Suppose now $f'(x_o) = 0$ at the point $x_o \in (a, b)$. In this case in a vicinity of the point $x = x_o$ the integrand does not oscillate.

Indeed, we can present $\exp(i\omega f)$ as follows

$$\exp(i\omega f) = \exp\left(i\omega[f(x_o) + f'(x_o)(x - x_o) + \cdots]\right)$$

and the oscillation period T can be estimated by the formula $T = 2\pi/(\omega f'(x_o))$.

Now, if $f'(x_o)$ is small, period T is large and, hence, there are no oscillations (practically!) in a vicinity of $x = x_o$.

Definition: A point $x = x_o$ is called the stationary point of a phase function $f(x)$ or, sometimes, the critical point of the integral I if $f'(x_o) = 0$.

Suppose further that I has only one critical point $x = x_o$ distant from both ends a and b.

To derive the asymptotics of I in this case we divide the integral into the following three parts $I = I_a + I_{c.p.} + I_b$, where

$$I_a = \int_a^{x_o - \Delta} ge^{i\omega f} dx\ , \quad I_{c.p.} = \int_{x_o - \Delta}^{x_o + \Delta} ge^{i\omega f} dx \quad \text{and } I_b = \int_{x_o + \Delta}^{b} ge^{i\omega f} dx\ ,$$

and Δ is an auxiliary parameter that separates an interval with the stationary point.

A.2 Contribution of the critical point

Consider the leading term of the asymptotics coming out from $I_{c.p.}$. In a vicinity of x_o we expand $f(x)$ and $g(x)$ in power series

$$\begin{aligned} f(x) &= f(x_o) + \frac{1}{2}f''(x_o)(x-x_o)^2 + \cdots \ , \\ g(x) &= g(x_o) + \cdots \ , \end{aligned}$$

so that now

$$\begin{aligned} I_{c.p.} &= \int_{x_o+\Delta}^{x_o-\Delta} \big(g(x_o)+\cdots\big) e^{i\omega[f(x_o)+\frac{1}{2}f''(x_o)(x-x_o)^2+\cdots]} dx \simeq \\ &\simeq g(x_o)e^{i\omega f(x_o)} \int_{x_o-\Delta}^{x_o+\Delta} e^{i\frac{\omega}{2}f''(x_o)(x-x_o)^2} dx + \cdots \ . \end{aligned}$$

In order to get the main term of the asymptotics we have to take into account only the first item in the latter expression.

Let us present a second derivative of the function $f(x)$ in the form

$$f''(x_o) = \text{sgn}\ \big(f''(x_o)\big) \quad \big|f''(x_o)\big| \ .$$

Then we develop the integral

$$\int_{x_o-\Delta}^{x_o+\Delta} \exp\left[\frac{i\omega}{2}f''(x_o)(x-x_o)^2\right] dx$$

as follows

$$\begin{aligned} &\int_{x_o-\Delta}^{x_o+\Delta} \exp\left[i\frac{\omega}{2}\text{sgn}\ (f^{''}(x_o))|f''(x_o)|(x-x_o)^2\right] dx = \\ &= \sqrt{\frac{2}{\omega|f^{''}(x_o)|}} \int_{-\Delta_1}^{\Delta_1} \exp\left[i\ \text{sgn}\ (f^{''}(x_o))\xi^2\right] d\xi \ , \end{aligned}$$

where ξ is related to x by the formula $\xi = \sqrt{(\omega/2)|f''(x_o)|}(x-x_o)$. A new limit of integration Δ_1 is given by the expression $\Delta_1 = \sqrt{(\omega/2)|f''(x_o)|}\Delta$. Note that Δ_1 contains the large parameter ω and tends to infinity as $\omega \to \infty$.

Let us introduce a new variable ρ by the formula

$$\xi = \rho \exp\left[i\frac{\pi}{4}\ \text{sgn}\ (f^{''}(x_o)\right] \ ,$$

then

$$\int_{-\Delta_1}^{\Delta_1} \exp\left[i\ \text{sgn}\ \big(f^{''}(x_o)\big)\ \xi^2\right] d\xi = \exp\left[i\frac{\pi}{4}\ \text{sgn}\ \big(f^{''}(x_o)\big)\right] \int_{-\Delta_2}^{\Delta_2} e^{-\rho^2} d\rho$$

where this time

$$\Delta_2 = \Delta_1 \exp\left[-i\frac{\pi}{4}\ \mathrm{sgn}\ \left(f^{''}(x_o)\right)\right] .$$

The last step in the development of the integral consists in extending the integral $\int_{-\Delta_2}^{\Delta_2} e^{-\rho^2} d\rho$ over the whole real axis, i.e. over $(-\infty, +\infty)$, obtaining

$$\int_{-\infty}^{+\infty} e^{-\rho^2} d\rho = \sqrt{\pi} .$$

Now by gathering all auxiliary results we obtain the main term of the contribution of the stationary point $x = x_o$

$$I_{c.p.} \cong g(x_o) e^{i\omega f(x_o)} \sqrt{\frac{2\pi}{\omega |f''(x_o)|}} \quad e^{i(\pi/4)\ \mathrm{sgn}\ (f^{''}(x_o))} .$$

The subsequent term of the integral $I_{c.p.}$ will have a multiplier $1/\sqrt{\omega}$ (after introducing a new variable ξ instead of x !) and therefore it will be of order $O\,(1/\omega)$, i.e. $1/\sqrt{\omega}$ time less than the leading one.

The contributions of I_a and I_b can be obtained by means of integration in parts and will be of order $O\,(1/\omega)$ (see the beginning of this Appendix).

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